

RENORMALIZED ENERGY FOR DISLOCATIONS IN QUASI-CRYSTALS

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ABSTRACT. Anti-plane shear deformations of a hexagonal quasi-crystal with multiple screw dislocations are considered. Using a variational formulation, the elastic equilibrium is characterized via limit of minimizers of a core-regularized energy functional. A sharp estimate of the asymptotic energy when the core radius tends to zero is obtained using higher-order Γ -convergence. Also, the interaction between dislocations and the Peach-Köehler force at each dislocation are analyzed.

Keywords: dislocation; renormalized energy; Γ -convergence.

1. INTRODUCTION

1.1. Problem Settings. Quasi-crystals were introduced in 1982 by Shechtman(see [17]) as a kind of non-crystalline condensed matter state. In contrast with crystals with periodic atomic arrangement, quasi-crystals only exhibit quasi-periodicity, i.e. they have perfect long-range order (like mirror symmetry) but no three-dimensional periodicity.

Unlike many other amorphous solids, quasi-crystals have similar elastic properties to these of crystals. More importantly, based on the Landau density wave theory(see [9]), quasi-crystals can be described as a projection of higher-dimensional crystals into a lower-dimensional space. This requires two displacement fields \vec{u} and \vec{w} defined in the physical domain of the quasi-crystal, where \vec{u} is a phonon field which is similar to the displacement field in crystals and \vec{w} is an extra phase field. Also, we may define the strain and stress tensors in phonon space and phase space.

To be precise, we consider anti-plane shear deformations of a one-dimensional hexagonal quasi-crystal (see [5], [6], [7], [8], [9], [14]). Given an elastic body $\Xi = \Omega \times \mathbb{R}$, where $\Omega \subset \mathbb{R}^2$ is simply-connected, bounded and open, with Lipschitz $\partial\Omega$, we denote the phonon deformation as

$$\Phi : (x, y, z) \rightarrow (x, y, z + u(x, y)),$$

and the phase deformation as

$$\Psi : (x, y, z) \rightarrow (x, y, z + w(x, y)),$$

for some functions $u, w : \Omega \rightarrow \mathbb{R}$. This allows us to reduce the three-dimensional problem to a two-dimensional setting. Hence, the phonon strain tensor is defined as

$$U := \nabla(0, 0, u) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & 0 \end{pmatrix}, \quad (1.1)$$

which can be symmetrized as

$$\tilde{U} := \frac{U + U^T}{2} = \begin{pmatrix} 0 & 0 & \frac{1}{2} \frac{\partial u}{\partial x} \\ 0 & 0 & \frac{1}{2} \frac{\partial u}{\partial y} \\ \frac{1}{2} \frac{\partial u}{\partial x} & \frac{1}{2} \frac{\partial u}{\partial y} & 0 \end{pmatrix},$$

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and the non-symmetric phase strain tensor is defined as

$$W := \nabla(0, 0, w) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & 0 \end{pmatrix}. \quad (1.2)$$

The relations (1.1) and (1.2) hold for a quasi-crystal when dislocations are absent. If dislocations are taken into consideration, then the strain tensor is singular at the site of the dislocations, and in particular it is a line singularity for a screw dislocation. Dislocations are one-dimensional defects in a crystalline-type material, whose presence may greatly affect the elastic and other properties (see [11] and [15]). Dislocation lines of quasi-crystals were observed in experiments soon after Shechtman's discover (see [1], [12], [13], [14]).

In a quasi-crystal undergoing a shear deformation, a screw dislocation may be described by a position $(x, y) \in \Omega$ and a Burger's vector $\vec{b} = b\vec{e}_z$. Here \vec{e}_z denotes the unit vector in the z direction and b , the Burger's modulus, represents the magnitude of the dislocation. The presence of dislocation yields a singularity at position (x, y) and thus strain tensors fail to be the gradients of smooth displacement fields, i.e. (1.1) and (1.2) do not hold any more.

To be precise, consider N dislocations at $\vec{d}_i = (x_i, y_i)$ for $i = 1, 2, \dots, N$, with Burger's vector for the phonon field given by $\vec{b}_u^i = b_u^i \vec{e}_z$ and for the phase field given by $\vec{b}_w^i = b_w^i \vec{e}_z$. The strain tensors U and W now satisfy

$$(\nabla \times U) \cdot \vec{e}_z = \sum_{i=1}^N \vec{b}_u^i \delta_{\vec{d}_i}, \quad (\nabla \times W) \cdot \vec{e}_z = \sum_{i=1}^N \vec{b}_w^i \delta_{\vec{d}_i},$$

which is equivalent to

$$b_u^i = \int_{\ell_i} U \cdot t ds, \quad b_w^i = \int_{\ell_i} W \cdot t ds,$$

where ℓ_i is any counterclockwise loop that surrounds \vec{d}_i and no other dislocation points, t is the tangent of ℓ_i and ds is the line differential. Similarly, we can still define the symmetrized phonon strain tensor $\tilde{U} = \frac{U + U^T}{2}$.

Denote the phonon stress tensor as σ and the phase stress tensor as ρ , which are 3×3 matrices in principle. For the convenience of computation, we may straighten σ , ρ , \tilde{U} and W to column vectors with 9 components. Then the generalized Hooke's law (see [9]) reads as

$$\begin{pmatrix} \sigma \\ \rho \end{pmatrix} = \begin{pmatrix} \mathcal{C} & \mathcal{R} \\ \mathcal{R}^T & \mathcal{K} \end{pmatrix} \begin{pmatrix} \tilde{U} \\ W \end{pmatrix},$$

where \mathcal{C} , \mathcal{R} , \mathcal{K} are 9×9 matrices such that $\begin{pmatrix} \mathcal{C} & \mathcal{R} \\ \mathcal{R}^T & \mathcal{K} \end{pmatrix}$ is positive definite and depends on the species of the quasi-crystal. The equilibrium equations are

$$\nabla \cdot \sigma = 0, \quad \nabla \cdot \rho = 0,$$

where the divergence is performed row by row. Here we use straightened vectors and matrices interchangeably. The free energy is

$$J[U, W] := \int_{\Xi} \mathfrak{F}[U, W] dx dy dz,$$

where the energy density \mathfrak{F} is given by

$$\mathfrak{F}[U, W] := \frac{1}{2} \begin{pmatrix} \tilde{U}^T & W^T \end{pmatrix} \begin{pmatrix} \mathcal{C} & \mathcal{R} \\ \mathcal{R}^T & \mathcal{K} \end{pmatrix} \begin{pmatrix} \tilde{U} \\ W \end{pmatrix}.$$

We intend to study the structure of the energy associated with this system.

1.2. Problem Simplification. Since U and W are sparse matrices, we can reduce the 18-variable problem to a 4-variable problem (see [9]). In particular, for N dislocation points at \vec{d}_i , $i = 1, 2, \dots, N$, with Burger's vectors for the phonon field given by \vec{b}_u^i and for the phase field given by \vec{b}_w^i , it suffices to consider $\mathcal{U} = (\mathcal{U}_x, \mathcal{U}_y)$ and $\mathcal{W} = (\mathcal{W}_x, \mathcal{W}_y)$ satisfying

$$\begin{cases} \begin{pmatrix} \sigma \\ \rho \end{pmatrix} = \begin{pmatrix} \mathcal{C} & \mathcal{R} \\ \mathcal{R}^T & \mathcal{K} \end{pmatrix} \begin{pmatrix} \mathcal{U} \\ \mathcal{W} \end{pmatrix}, \\ \nabla \times \mathcal{U} = \sum_{i=1}^N b_u^i \delta_{\vec{d}_i}, \quad \nabla \times \mathcal{W} = \sum_{i=1}^N b_w^i \delta_{\vec{d}_i}, \\ \nabla \cdot \sigma = 0, \quad \nabla \cdot \rho = 0, \end{cases} \quad (1.3)$$

where $\sigma = (\sigma_x, \sigma_y)$, $\rho = (\rho_x, \rho_y)$ are vectors with 2 components, and \mathcal{C} , \mathcal{R} , \mathcal{K} are 2×2 matrices, \mathcal{C} and \mathcal{K} are symmetric and positive definite, $\nabla \cdot \vec{f} := \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}$ and $\nabla \times \vec{f} := \frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}$. Roughly speaking, \mathcal{U} plays the role of $\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)$ and \mathcal{W} plays the role of $\left(\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}\right)$. Here we omit the symmetrization procedure of \mathcal{U} since it can be directly incorporated into Hooke's law, and we do not change the notation for σ , ρ , \mathcal{C} , \mathcal{R} , \mathcal{K} . The free energy is

$$J[\mathcal{U}, \mathcal{W}] := \int_{\Omega} \mathfrak{F}[\mathcal{U}, \mathcal{W}] dx dy,$$

with density

$$\mathfrak{F}[\mathcal{U}, \mathcal{W}] := \frac{1}{2} \begin{pmatrix} \mathcal{U}^T & \mathcal{W}^T \end{pmatrix} \begin{pmatrix} \mathcal{C} & \mathcal{R} \\ \mathcal{R}^T & \mathcal{K} \end{pmatrix} \begin{pmatrix} \mathcal{U} \\ \mathcal{W} \end{pmatrix}.$$

In a hexagonal quasi-crystal (see [9]), we may further simplify the Hooke's law as

$$\mathcal{C} = \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}, \quad \mathcal{R} = \mathcal{R}^T = \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}, \quad \mathcal{K} = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix},$$

for some constants C , R , K with

$$C, K > 0 \text{ and } CK > R^2, \quad (1.4)$$

i.e. the matrix $\begin{pmatrix} \mathcal{C} & \mathcal{R} \\ \mathcal{R}^T & \mathcal{K} \end{pmatrix}$ is positive definite. Also, the free energy density reduces to

$$\mathfrak{F}[\mathcal{U}, \mathcal{W}] = \frac{1}{2} \left(C |\mathcal{U}|^2 + K |\mathcal{W}|^2 + 2R(\mathcal{U} \cdot \mathcal{W}) \right). \quad (1.5)$$

1.3. Core Regularization. It is well-known that in a neighborhood of a dislocation point, the free energy blows up (see [7] and [8]). Similar to the techniques in [6] and [8] for crystals, we consider a variational formulation by removing a core $B_\epsilon(\vec{d}_i) = \{\vec{d} = (x, y) : |\vec{d} - \vec{d}_i| \leq \epsilon\}$ around each dislocation, and we consider the minimization problem

$$\min_{(\mathfrak{U}, \mathfrak{W}) \in H_0^\epsilon} \int_{\Omega_\epsilon} \mathfrak{F}[\mathfrak{U}, \mathfrak{W}] dx dy, \quad (1.6)$$

where $\Omega_\epsilon := \Omega \setminus \left(\bigcup_{i=1}^N B_\epsilon(\vec{d}_i) \right)$ and the admissible set is defined by

$$H_0^\epsilon = \left\{ (\mathfrak{U}, \mathfrak{W}) : \mathfrak{U}, \mathfrak{W} \in L^2(\Omega_\epsilon), \quad \nabla \times \mathfrak{U} = 0, \nabla \times \mathfrak{W} = 0 \text{ in } \Omega_\epsilon, \right. \\ \left. \int_{\partial B_\epsilon(\vec{d}_i)} \mathfrak{U} \cdot t ds = b_u^i, \quad \int_{\partial B_\epsilon(\vec{d}_i)} \mathfrak{W} \cdot t ds = b_w^i, \quad i = 1, 2, \dots, N \right\},$$

where t is the unit tangent vector at $\partial B_\epsilon(\vec{d}_i)$. Here $\mathfrak{U} \cdot t$ and $\mathfrak{W} \cdot t$ are the tangential traces of \mathfrak{U} and \mathfrak{W} , which are well-defined in the L^2 curl-free space (see [7] and [8]).

Assume that the solution to the above minimization problem admits a unique solution as $(\mathcal{U}_\epsilon, \mathcal{W}_\epsilon)$. Our goal is to study the behavior of $(\mathcal{U}_\epsilon, \mathcal{W}_\epsilon)$ and of the free energy

$$J_\epsilon[\mathcal{U}_\epsilon, \mathcal{W}_\epsilon] := \int_{\Omega_\epsilon} \mathfrak{F}[\mathcal{U}_\epsilon, \mathcal{W}_\epsilon] dx dy,$$

as $\epsilon \rightarrow 0$.

1.4. Main Theorem. We intend to use Γ -convergence to analyze the minimizer and energy structure. Define the functional $J_\epsilon^{(0)} : L^2(\Omega) \times L^2(\Omega) \rightarrow [0, \infty]$ by

$$J_\epsilon^{(0)}[\mathfrak{U}_\epsilon, \mathfrak{W}_\epsilon] := \begin{cases} \int_{\Omega_\epsilon} \frac{1}{2} \left(C |\mathfrak{U}_\epsilon|^2 + K |\mathfrak{W}_\epsilon|^2 + 2R(\mathfrak{U}_\epsilon \cdot \mathfrak{W}_\epsilon) \right) dx dy \\ \text{if } (\mathfrak{U}_\epsilon, \mathfrak{W}_\epsilon) = \left(\frac{\tilde{\mathfrak{U}}_\epsilon}{|\ln(\epsilon)|^{1/2}}, \frac{\tilde{\mathfrak{W}}_\epsilon}{|\ln(\epsilon)|^{1/2}} \right) \text{ for some } (\tilde{\mathfrak{U}}_\epsilon, \tilde{\mathfrak{W}}_\epsilon) \in H_0^\epsilon, \\ \infty \text{ otherwise in } L^2(\Omega) \times L^2(\Omega). \end{cases}$$

Theorem 1.1. (Compactness)(see Section 3.1) Assume that (1.4) holds and $(\mathfrak{U}_\epsilon, \mathfrak{W}_\epsilon) \in L^2(\Omega) \times L^2(\Omega)$ satisfy

$$\sup_{\epsilon > 0} J_\epsilon^{(0)}[\mathfrak{U}_\epsilon, \mathfrak{W}_\epsilon] \leq C_0.$$

Then there exists $v_u, v_w \in H^1(\Omega)$ such that up to the extraction of subsequence (non-relabelled),

$$(\mathbf{1}_{\Omega_\epsilon} \mathfrak{U}_\epsilon, \mathbf{1}_{\Omega_\epsilon} \mathfrak{W}_\epsilon) \rightharpoonup (\nabla v_u, \nabla v_w) \text{ in weak-} L^2 \text{ as } \epsilon \rightarrow 0.$$

With compactness theorem in hand, we can show the zeroth-order Γ -convergence.

Theorem 1.2. (0^{th} -Order Γ -Convergence)(see Section 3.2) Assume that (1.4) holds. Define the functional $J_0^{(0)} : L^2(\Omega) \times L^2(\Omega) \rightarrow [0, \infty]$ as

$$J_0^{(0)}[\mathfrak{U}, \mathfrak{W}] := \begin{cases} \int_{\Omega} \frac{1}{2} \left(C |\nabla v_u|^2 + K |\nabla v_w|^2 + 2R(\nabla v_u \cdot \nabla v_w) \right) + \sum_{i=1}^N \frac{C(b_u^i)^2 + K(b_w^i)^2 + 2R(b_u^i)(b_w^i)}{4\pi} \\ \text{if } (\mathfrak{U}, \mathfrak{W}) = (\nabla v_u, \nabla v_w) \text{ for some } v_u, v_w \in H^1(\Omega), \\ \infty \text{ otherwise in } L^2(\Omega) \times L^2(\Omega). \end{cases}$$

Then

- (1) For any sequence of pairs $(\mathfrak{U}_\epsilon, \mathfrak{W}_\epsilon) \in L^2(\Omega) \times L^2(\Omega)$ such that $(\mathfrak{U}_\epsilon, \mathfrak{W}_\epsilon) \rightharpoonup (\mathfrak{U}, \mathfrak{W})$ in weak- $L^2(\Omega)$, we have $\liminf_{\epsilon \rightarrow 0} J_\epsilon^{(0)}[\mathfrak{U}_\epsilon, \mathfrak{W}_\epsilon] \geq J_0^{(0)}[\nabla v_u, \nabla v_w]$.
- (2) There exists a sequence of pairs $(\mathfrak{U}_\epsilon, \mathfrak{W}_\epsilon) \in L^2(\Omega) \times L^2(\Omega)$ such that $(\mathfrak{U}_\epsilon, \mathfrak{W}_\epsilon) \rightharpoonup (\mathfrak{U}, \mathfrak{W})$ in weak- $L^2(\Omega)$, we have $\limsup_{\epsilon \rightarrow 0} J_\epsilon^{(0)}[\mathfrak{U}_\epsilon, \mathfrak{W}_\epsilon] \leq J_0^{(0)}[\nabla v_u, \nabla v_w]$,

which means

$$J_\epsilon^{(0)}[\mathfrak{U}_\epsilon, \mathfrak{W}_\epsilon] \rightarrow J_0^{(0)}[\mathfrak{U}, \mathfrak{W}],$$

in the sense of Γ -convergence in weak- $L^2(\Omega)$

Γ -convergence naturally yields the convergence of minimum of energy functionals.

Corollary 1.3. (Core Energy)(see Section 3.2) Assume that (1.4) holds. We have

$$\inf_{\mathfrak{U}, \mathfrak{W}} J_0^{(0)}[\mathfrak{U}, \mathfrak{W}] = \sum_{i=1}^N \frac{C(b_u^i)^2 + K(b_w^i)^2 + 2R(b_u^i)(b_w^i)}{4\pi}.$$

Assume $(\mathcal{U}'_\epsilon, \mathcal{W}'_\epsilon)$ is the minimizer of $J_\epsilon^{(0)}$, then we have

$$J_\epsilon^{(0)}[\mathcal{U}'_\epsilon, \mathcal{W}'_\epsilon] = E_0 + o(1),$$

where the rescaled leading-order energy

$$E_0 = \sum_{i=1}^N \frac{C(b_u^i)^2 + K(b_w^i)^2 + 2R(b_u^i)(b_w^i)}{4\pi}. \quad (1.7)$$

The zeroth-order Γ -convergence result tells us the asymptotic behavior of leading-order free energy. However, the rescaling in $J_\epsilon^{(0)}$ suppress $O(1)$ term in the energy. As [3] revealed, more detailed information can be discovered when we get rid of the rescaling and go to first-order Γ -convergence. Define the functional $J_\epsilon^{(1)} : L^2(\Omega) \times L^2(\Omega) \rightarrow [0, \infty]$ as

$$J_\epsilon^{(1)} [\tilde{\mathfrak{U}}_\epsilon, \tilde{\mathfrak{W}}_\epsilon] := \begin{cases} \int_{\Omega_\epsilon} \frac{1}{2} \left(C |\tilde{\mathfrak{U}}_\epsilon|^2 + K |\tilde{\mathfrak{W}}_\epsilon|^2 + 2R(\tilde{\mathfrak{U}}_\epsilon \cdot \tilde{\mathfrak{W}}_\epsilon) \right) dx dy - |\ln(\epsilon)| \inf_{\tilde{\mathfrak{U}}, \tilde{\mathfrak{W}}} J_0^{(0)} [\tilde{\mathfrak{U}}, \tilde{\mathfrak{W}}] & \text{if } (\tilde{\mathfrak{U}}_\epsilon, \tilde{\mathfrak{W}}_\epsilon) \in H_{0,\epsilon}^\epsilon, \\ \infty & \text{otherwise in } L^2(\Omega) \times L^2(\Omega). \end{cases}$$

Theorem 1.4. (*1st-Order Γ -Convergence*) (see Section 3.3) Assume that (1.4) holds. Define the functional $J_0^{(1)} : L^2(\Omega) \times L^2(\Omega) \rightarrow [0, \infty]$ as

$$J_0^{(1)} [\tilde{\mathfrak{U}}, \tilde{\mathfrak{W}}] := \begin{cases} E_{\text{self}} + E_{\text{int}} + E_{\text{elastic}} & \text{if } (\tilde{\mathfrak{U}}, \tilde{\mathfrak{W}}) = \left(\nabla v_u + \sum_{i=1}^N \mathcal{U}_i, \nabla v_w + \sum_{i=1}^N \mathcal{W}_i \right), \\ & \text{for some } (v_u, v_w) \in H^1(\Omega), \\ \infty & \text{otherwise in } L^2(\Omega) \times L^2(\Omega), \end{cases}$$

where

$$\begin{aligned} E_{\text{self}} &:= \sum_{i=1}^N \int_{\Omega \setminus B_r(\vec{d}_i)} \frac{1}{2} \left(C |\mathcal{U}_i|^2 + K |\mathcal{W}_i|^2 + 2R(\mathcal{U}_i \cdot \mathcal{W}_i) \right) dx dy \\ &\quad + \sum_{i=1}^N \frac{(C(b_u^i)^2 + K(b_w^i)^2 + 2R(b_u^i)(b_w^i))}{4\pi} \ln(r), \\ E_{\text{int}} &:= \sum_{i=1}^{N-1} \sum_{j=i}^N \int_{\Omega} \left(C(\mathcal{U}_i \cdot \mathcal{U}_j) + K(\mathcal{W}_i \cdot \mathcal{W}_j) + R(\mathcal{U}_i \cdot \mathcal{W}_j) + R(\mathcal{W}_j \cdot \mathcal{U}_i) \right), \\ E_{\text{elastic}} &:= J[\nabla v_u, \nabla v_w] + \sum_{i=1}^N \int_{\partial\Omega} \left(v_u(C\mathcal{U}_i + R\mathcal{W}_i) + v_w(K\mathcal{W}_i + R\mathcal{U}_i) \right) \cdot nds. \end{aligned}$$

Then

- (1) For any sequence of pairs $(\tilde{\mathfrak{U}}_\epsilon, \tilde{\mathfrak{W}}_\epsilon) \in L^2(\Omega) \times L^2(\Omega)$ such that $(\tilde{\mathfrak{U}}_\epsilon, \tilde{\mathfrak{W}}_\epsilon) \rightharpoonup (\tilde{\mathfrak{U}}, \tilde{\mathfrak{W}})$ in weak- $L^2(\Omega)$, we have $\liminf_{\epsilon \rightarrow 0} J_\epsilon^{(1)} [\tilde{\mathfrak{U}}_\epsilon, \tilde{\mathfrak{W}}_\epsilon] \geq J_0^{(1)} [\nabla v_u, \nabla v_w]$.
- (2) There exists a sequence of pairs $(\tilde{\mathfrak{U}}_\epsilon, \tilde{\mathfrak{W}}_\epsilon) \in L^2(\Omega) \times L^2(\Omega)$ such that $(\tilde{\mathfrak{U}}_\epsilon, \tilde{\mathfrak{W}}_\epsilon) \rightharpoonup (\tilde{\mathfrak{U}}, \tilde{\mathfrak{W}})$ in weak- $L^2(\Omega)$, we have $\limsup_{\epsilon \rightarrow 0} J_\epsilon^{(1)} [\tilde{\mathfrak{U}}_\epsilon, \tilde{\mathfrak{W}}_\epsilon] \leq J_0^{(1)} [\nabla v_u, \nabla v_w]$,

which means

$$J_\epsilon^{(1)} [\tilde{\mathfrak{U}}_\epsilon, \tilde{\mathfrak{W}}_\epsilon] \rightarrow J_0^{(1)} [\tilde{\mathfrak{U}}, \tilde{\mathfrak{W}}],$$

in the sense of Γ -convergence in weak- $L^2(\Omega)$.

Similarly, we have a better approximation of energy functionals.

Corollary 1.5. (*Renormalized Energy*) (see Section 3.3) Assume that (1.4) holds. We have

$$\inf_{\tilde{\mathfrak{U}}, \tilde{\mathfrak{W}}} J_0^{(1)} [\tilde{\mathfrak{U}}, \tilde{\mathfrak{W}}] = F_{\text{self}} + F_{\text{int}} + F_{\text{elastic}},$$

where

$$\begin{aligned}
F_{\text{self}} &:= \sum_{i=1}^N \int_{\Omega \setminus B_r(\vec{d}_i)} \frac{1}{2} \left(C |\mathcal{U}_i|^2 + K |\mathcal{W}_i|^2 + 2R(\mathcal{U}_i \cdot \mathcal{W}_i) \right) dx dy \\
&\quad + \sum_{i=1}^N \frac{(C(b_u^i)^2 + K(b_w^i)^2 + 2R(b_u^i)(b_w^i))}{4\pi} \ln(r), \\
F_{\text{int}} &:= \sum_{i=1}^{N-1} \sum_{j=i}^N \int_{\Omega} \left(C(\mathcal{U}_i \cdot \mathcal{U}_j) + K(\mathcal{W}_i \cdot \mathcal{W}_j) + R(\mathcal{U}_i \cdot \mathcal{W}_j) + R(\mathcal{U}_j \cdot \mathcal{W}_i) \right), \\
F_{\text{elastic}} &:= J[\nabla u_0, \nabla w_0] + \sum_{i=1}^N \int_{\partial\Omega} \left(u_0(C\mathcal{U}_i + R\mathcal{W}_i) + w_0(K\mathcal{W}_i + R\mathcal{U}_i) \right) \cdot n ds,
\end{aligned} \tag{1.8}$$

in which (u_0, w_0) is the minimizer of

$$I[v_u, v_w] = J[\nabla v_u, \nabla v_w] + \sum_{i=1}^N \int_{\partial\Omega} \left(v_u(C\mathcal{U}_i + R\mathcal{W}_i) + v_w(K\mathcal{W}_i + R\mathcal{U}_i) \right) \cdot n ds.$$

Assume $(\tilde{\mathcal{U}}'_\epsilon, \tilde{\mathcal{W}}'_\epsilon) \in H_0^\epsilon$ is the minimizer of $J_\epsilon^{(1)}$, then we have

$$J_\epsilon^{(1)}[\tilde{\mathcal{U}}'_\epsilon, \tilde{\mathcal{W}}'_\epsilon] = F_{\text{self}} + F_{\text{int}} + F_{\text{elastic}} + o(1).$$

As corollaries, we can now state a characterization of the structure of minimizer $(\mathcal{U}_\epsilon, \mathcal{W}_\epsilon)$ and energy $J_\epsilon[\mathcal{U}_\epsilon, \mathcal{W}_\epsilon]$ in (1.6).

Theorem 1.6. (Minimizer Structure)(see Section 3.4) Assume that (1.4) holds. The problem (1.6) admits a unique solution

$$\mathcal{U}_\epsilon = \sum_{i=1}^N \mathcal{U}_i + \nabla u_\epsilon, \quad \mathcal{W}_\epsilon = \sum_{i=1}^N \mathcal{W}_i + \nabla w_\epsilon,$$

where

$$\begin{aligned}
\mathcal{U}_i &= \frac{b_u^i}{2\pi} \frac{1}{(x - x_i)^2 + (y - y_i)^2} \begin{pmatrix} -(y - y_i) \\ (x - x_i) \end{pmatrix}, \\
\mathcal{W}_i &= \frac{b_w^i}{2\pi} \frac{1}{(x - x_i)^2 + (y - y_i)^2} \begin{pmatrix} -(y - y_i) \\ (x - x_i) \end{pmatrix},
\end{aligned}$$

and (u_ϵ, w_ϵ) is the unique minimizer of

$$\begin{aligned}
I_\epsilon[u_\epsilon, w_\epsilon] &:= J_\epsilon[\nabla u_\epsilon, \nabla w_\epsilon] + \sum_{i=1}^N \int_{\partial\Omega} \left(u_\epsilon(C\mathcal{U}_i + R\mathcal{W}_i) + w_\epsilon(K\mathcal{W}_i + R\mathcal{U}_i) \right) \cdot n ds \\
&\quad - \sum_{i=1}^N \sum_{j \neq i} \int_{\partial B_\epsilon(x_i, y_i)} \left(u_\epsilon(C\mathcal{U}_j + R\mathcal{W}_j) + w_\epsilon(K\mathcal{W}_j + R\mathcal{U}_j) \right) \cdot n ds,
\end{aligned}$$

subject to $\int_B u_\epsilon dx dy = 0$ and $\int_B w_\epsilon dx dy = 0$ for some ball $B \subset \Omega_\epsilon$, with n the outward unit normal vector on $\partial\Omega$.

Furthermore, $(\mathcal{U}_\epsilon, \mathcal{W}_\epsilon)$ converges in weak- $L^2(\Omega)$ as $\epsilon \rightarrow 0$ to $(\mathcal{U}_0, \mathcal{W}_0)$ where

$$\mathcal{U}_0 = \sum_{i=1}^N \mathcal{U}_i + \nabla u_0, \quad \mathcal{W}_0 = \sum_{i=1}^N \mathcal{W}_i + \nabla w_0.$$

and $[u_0, w_0]$ is the unique minimizer of

$$I_0[u_0, w_0] = J[\nabla u_0, \nabla w_0] + \sum_{i=1}^N \int_{\partial\Omega} \left(u_0(C\mathcal{U}_i + R\mathcal{W}_i) + w_0(K\mathcal{W}_i + R\mathcal{U}_i) \right) \cdot n ds,$$

subject to $\int_B u_0 dx dy = 0$ and $\int_B w_0 dx dy = 0$ for some ball $B \subset \Omega_\epsilon$

Theorem 1.7. (Energy Structure)(see Section 3.4) Assume that (1.4) holds. We have

$$J_\epsilon[\mathcal{U}_\epsilon, \mathcal{W}_\epsilon] = \int_{\Omega_\epsilon} \mathfrak{F}[\mathcal{U}_\epsilon, \mathcal{W}_\epsilon] dx dy = E_0 \ln\left(\frac{1}{\epsilon}\right) + F + o(1),$$

where the core energy E_0 is defined in (1.7) and the renormalized energy $F = F_{\text{self}} + F_{\text{int}} + F_{\text{elastic}}$ is defined in (1.8).

Remark 1.8. The core energy is a leading singular term of $O(|\ln(\epsilon)|)$, which confirms that the free energy is not finite when dislocations are present. The $O(1)$ term F is usually called the renormalized energy and is physically meaningful. This type of asymptotic expansion was first derived for Ginzburg-Landau vortices in [4], and extended to the context of dislocation in [8]. The techniques to prove Γ -convergence results were first introduced in the study of the Ginzburg-Landau vortices (see [2] and [16]).

Note that the renormalized energy is independent of the radius ϵ and thus fully characterizes the energy structure around dislocations.

As an application of the energy structure, we prove that the interaction energy F_{int} obeys the inverse logarithmical law of the distance between two dislocations.

Theorem 1.9. (Interaction Energy)(see Section 4.1) Assume that (1.4) holds. We have

$$F_{\text{int}} = \sum_{i=1}^{N-1} \sum_{j=i}^N \frac{C b_u^i b_u^j + K b_w^i b_w^j + R b_u^i b_w^j + R b_w^i b_u^j}{2\pi} \ln\left(\frac{1}{|\vec{d}_i - \vec{d}_j|}\right) + O(1).$$

When multiple dislocations are present, defects interact with themselves by means of the so-called Peach-Köhler force, which is defined as the negative gradient of renormalized energy F at the dislocation points (see [10]).

Theorem 1.10. (Peach-Köhler force)(see Section 4.2) Assume that (1.4) holds. The Peach-Köhler force acting at \vec{d}_k is given by

$$\nabla_{\vec{d}_k} F = - \int_{\partial B_r(\vec{d}_k)} \left(\mathfrak{F}[\mathcal{U}_0, \mathcal{W}_0] \mathbf{1} - (C \mathcal{U}_0 \otimes \mathcal{U}_0 + K \mathcal{W}_0 \otimes \mathcal{W}_0 + R \mathcal{U}_0 \otimes \mathcal{W}_0 + R \mathcal{W}_0 \otimes \mathcal{U}_0) \right) \cdot n ds,$$

for $r < \frac{1}{2} \min_k \left(\text{dist}(\vec{d}_k, \partial\Omega) \right)$.

Remark 1.11. The integrand in Theorem 1.10

$$E = - \left(\mathfrak{F}[\mathcal{U}_0, \mathcal{W}_0] \mathbf{1} - (C \mathcal{U}_0 \otimes \mathcal{U}_0 + K \mathcal{W}_0 \otimes \mathcal{W}_0 + R \mathcal{U}_0 \otimes \mathcal{W}_0 + R \mathcal{W}_0 \otimes \mathcal{U}_0) \right).$$

is usually called the Eshelby stress tensor.

Our paper is organized as follows: in Section 2 we present some preliminary results on the minimization problem (1.6) of J_ϵ for fixed ϵ ; in Section 3 we derive the zeroth-order and first-order Γ -convergence of the free energy when $\epsilon \rightarrow 0$ and study the structure of minimizer and energy; Finally, in Section 4 we introduce two applications of the renormalized energy: the interaction between dislocations and the Peach-Köhler force.

2. PRELIMINARIES

In this section, we consider the minimization problem (1.6) of J_ϵ for fixed ϵ .

2.1. Euler-Lagrange Equation. We start with the equations that minimizer of J_ϵ should satisfy and the uniqueness of minimizer.

Lemma 2.1. *Assume that (1.4) holds and $(\mathcal{U}_\epsilon, \mathcal{W}_\epsilon)$ is the minimizer of J_ϵ in $H_0^\epsilon(\Omega)$. Then it satisfies the equations*

$$\begin{cases} \nabla \cdot (\mathcal{C}\mathcal{U}_\epsilon + \mathcal{R}\mathcal{W}_\epsilon) = \nabla \cdot (\mathcal{K}\mathcal{W}_\epsilon + \mathcal{R}^T\mathcal{U}_\epsilon) &= 0 \text{ in } \Omega_\epsilon, \\ (\mathcal{C}\mathcal{U}_\epsilon + \mathcal{R}\mathcal{W}_\epsilon) \cdot n = (\mathcal{K}\mathcal{W}_\epsilon + \mathcal{R}^T\mathcal{U}_\epsilon) \cdot n &= 0 \text{ on } \partial\Omega_\epsilon, \end{cases} \quad (2.1)$$

where n is the outward normal vector to $\partial\Omega_\epsilon$. Moreover, the solution to (2.1) is unique.

Proof. The free energy density in Ω_ϵ is given by

$$\begin{aligned} \mathfrak{F}[\mathfrak{U}, \mathfrak{W}] &= \frac{1}{2} \begin{pmatrix} \mathfrak{U}^T & \mathfrak{W}^T \end{pmatrix} \begin{pmatrix} \mathcal{C} & \mathcal{R} \\ \mathcal{R}^T & \mathcal{K} \end{pmatrix} \begin{pmatrix} \mathfrak{U} \\ \mathfrak{W} \end{pmatrix} \\ &= \frac{1}{2} \left(\mathfrak{U}^T \mathcal{C} \mathfrak{U} + \mathfrak{W}^T \mathcal{K} \mathfrak{W} + \mathfrak{U}^T \mathcal{R} \mathfrak{W} + \mathfrak{W}^T \mathcal{R}^T \mathfrak{U} \right). \end{aligned}$$

For any $(\mathfrak{U}, \mathfrak{W})$ and $(\bar{\mathfrak{U}}, \bar{\mathfrak{W}})$ in H_0^ϵ , we must have $\mathfrak{U} - \bar{\mathfrak{U}} = \nabla P$ and $\mathfrak{W} - \bar{\mathfrak{W}} = \nabla Q$ for some $P, Q \in H^1(\Omega_\epsilon)$ due to curl-free condition. Hence, the first-order variation is

$$\begin{aligned} \delta J_\epsilon[\mathfrak{U}, \mathfrak{W}](p, q) &= \lim_{\theta \rightarrow 0} \frac{J_\epsilon[\mathfrak{U} + \theta \nabla p, \mathfrak{W} + \theta \nabla q] - J_\epsilon[\mathfrak{U}, \mathfrak{W}]}{\theta} \\ &= - \int_{\Omega_\epsilon} \left(p \nabla \cdot (\mathcal{C}\mathfrak{U} + \mathcal{R}\mathfrak{W}) + q \nabla \cdot (\mathcal{K}\mathfrak{W} + \mathcal{R}^T\mathfrak{U}) \right) dx dy \\ &\quad + \int_{\partial\Omega_\epsilon} \left(p(\mathcal{C}\mathfrak{U} + \mathcal{R}\mathfrak{W}) \cdot n + q(\mathcal{K}\mathfrak{W} + \mathcal{R}^T\mathfrak{U}) \cdot n \right) ds. \end{aligned}$$

Thus, setting $\delta J_\epsilon[\mathfrak{U}, \mathfrak{W}](p, q) = 0$ for any $p, q \in H^1(\Omega_\epsilon)$, we can deduce that the minimizer $(\mathcal{U}_\epsilon, \mathcal{W}_\epsilon)$ is a weak solution of the Euler-Lagrange equations (2.1).

To prove uniqueness, assume that $(\mathcal{U}_\epsilon, \mathcal{W}_\epsilon)$ and $(\bar{\mathcal{U}}_\epsilon, \bar{\mathcal{W}}_\epsilon)$ are two solutions to (2.1). The difference $(f, g) = (\mathcal{U}_\epsilon - \bar{\mathcal{U}}_\epsilon, \mathcal{W}_\epsilon - \bar{\mathcal{W}}_\epsilon)$ must be curl-free and has zero loop integral around $\partial B_\epsilon(\vec{d}_i)$. Therefore, we must have $(f, g) = (\nabla F, \nabla G)$ for some $F, G \in H^1(\Omega_\epsilon)$. Since F and G satisfy the equation

$$\int_{\Omega_\epsilon} \left((\nabla p)^T (\mathcal{C} \nabla F + \mathcal{R} \nabla G) + (\nabla q)^T (\mathcal{K} \nabla G + \mathcal{R}^T \nabla F) \right) dx dy = 0,$$

for any $p, q \in H^1(\Omega_\epsilon)$, taking $p = F$ and $q = G$, considering $\begin{pmatrix} \mathcal{C} & \mathcal{R} \\ \mathcal{R}^T & \mathcal{K} \end{pmatrix}$ is positive definite, we must have $\nabla F = \nabla G = 0$, and the uniqueness follows. \square

2.2. Estimate and Energy for Single Dislocation. In this section, we further restrict the discussion to the case in which $\Omega = B_r(\vec{d}_0)$ for constant $r \gg \epsilon$, with only one dislocation at $\vec{d}_0 = (x_0, y_0)$ with Burger's vector of phonon field as \vec{b}_u and of phase field as \vec{b}_w . Solving the above Euler-Lagrange equations (2.1), by a linear combination, we get

$$\begin{cases} \nabla \cdot \mathcal{U}_\epsilon = \nabla \cdot \mathcal{W}_\epsilon &= 0 \text{ in } \Omega_\epsilon, \\ \mathcal{U}_\epsilon \cdot n = \mathcal{W}_\epsilon \cdot n &= 0 \text{ on } \partial\Omega_\epsilon, \end{cases}$$

in $H_0^\epsilon(\Omega)$. Hence, there exists potential functions $U_\epsilon(x, y)$ and $W_\epsilon(x, y)$ such that $\nabla U_\epsilon = \mathcal{U}_\epsilon$, $\nabla W_\epsilon = \mathcal{W}_\epsilon$ and

$$\Delta U_\epsilon = \Delta W_\epsilon = 0 \text{ in } \Omega_\epsilon.$$

Therefore, we are lead to solving Laplace's equations in an annulus with Neumann boundary $\frac{\partial U_\epsilon}{\partial n} = \frac{\partial W_\epsilon}{\partial n} = 0$. This system has a unique solution subject to the normalization conditions $\int_{\partial B_\epsilon(\vec{d}_0)} dU_\epsilon = b_u$

and $\int_{\partial B_\epsilon(\vec{d}_0)} dW_\epsilon = b_w$, and we obtain the explicit solution as

$$U_\epsilon = \frac{b_u}{2\pi} \arctan\left(\frac{y-y_0}{x-x_0}\right), \quad W_\epsilon = \frac{b_w}{2\pi} \arctan\left(\frac{y-y_0}{x-x_0}\right) \quad \text{for } (x, y) \in \Omega_\epsilon.$$

Hence, we have

$$\mathcal{U}_\epsilon = \frac{b_u}{2\pi} \frac{1}{(x-x_0)^2 + (y-y_0)^2} \left(-(y-y_0), (x-x_0) \right), \quad (2.2)$$

$$\mathcal{W}_\epsilon = \frac{b_w}{2\pi} \frac{1}{(x-x_0)^2 + (y-y_0)^2} \left(-(y-y_0), (x-x_0) \right), \quad (2.3)$$

for $(x, y) \in \Omega_\epsilon$, and we note that these are independent of ϵ and r . Therefore, the minimum free energy can be obtained explicitly as

$$J_\epsilon = \int_{\Omega_\epsilon} \mathfrak{F}[\mathcal{U}_\epsilon, \mathcal{W}_\epsilon] dx dy = (Cb_u^2 + Kb_w^2 + 2Rb_ub_w) \frac{1}{4\pi} \ln\left(\frac{r}{\epsilon}\right).$$

2.3. Estimate and Energy for Multiple Dislocations. Now we consider the case with multiple dislocations in general domains. For fixed $\vec{d}_i = (x_i, y_i)$, assume that the single-dislocation solution is $(\mathcal{U}_i, \mathcal{W}_i)$. Based on analysis in Lemma 2.1, we must have

$$\mathcal{U}_\epsilon := \sum_{i=1}^N \mathcal{U}_i + \nabla u_\epsilon, \quad \mathcal{W}_\epsilon := \sum_{i=1}^N \mathcal{W}_i + \nabla w_\epsilon.$$

for some $u_\epsilon, w_\epsilon \in H^1(\Omega_\epsilon)$. We deduce

$$\begin{aligned} J_\epsilon[\mathcal{U}_\epsilon, \mathcal{W}_\epsilon] &= I_\epsilon[u_\epsilon, w_\epsilon] + \sum_{i=1}^N J_\epsilon[\mathcal{U}_i, \mathcal{W}_i] \\ &\quad + \sum_{i=1}^{N-1} \sum_{j=i}^N \int_{\Omega_\epsilon} \left(C(\mathcal{U}_i \cdot \mathcal{U}_j) + K(\mathcal{W}_i \cdot \mathcal{W}_j) + R(\mathcal{U}_i \cdot \mathcal{W}_j) + R(\mathcal{U}_j \cdot \mathcal{W}_i) \right), \end{aligned}$$

where

$$\begin{aligned} I_\epsilon[u_\epsilon, w_\epsilon] &:= J_\epsilon[\nabla u_\epsilon, \nabla w_\epsilon] + \sum_{i=1}^N \int_{\partial\Omega} \left(u_\epsilon(C\mathcal{U}_i + R\mathcal{W}_i) + w_\epsilon(K\mathcal{W}_i + R\mathcal{U}_i) \right) \cdot n ds \\ &\quad - \sum_{i=1}^N \sum_{j \neq i} \int_{\partial B_\epsilon(x_i, y_i)} \left(u_\epsilon(C\mathcal{U}_j + R\mathcal{W}_j) + w_\epsilon(K\mathcal{W}_j + R\mathcal{U}_j) \right) \cdot n ds. \end{aligned}$$

Therefore, in order to minimize J_ϵ , it suffices to consider the problem:

(M_ϵ) : Minimize $I_\epsilon[\mathbf{u}, \mathbf{w}]$ for $\mathbf{u}, \mathbf{w} \in H^1(\Omega_\epsilon)$ subject to $\int_B \mathbf{u} dx dy = 0$ and $\int_B \mathbf{w} dx dy = 0$ for some ball $B \subset \Omega_\epsilon$, i.e. find the solution of

$$\min_{\mathbf{u}, \mathbf{w} \in H^1(\Omega_\epsilon)} I_\epsilon[\mathbf{u}, \mathbf{w}]. \quad (2.4)$$

This normalization is for the convenience of coercivity and will not affect the minimizing process since adding a constant to \mathbf{u} or \mathbf{w} will not affect the value of $I_\epsilon[\mathbf{u}, \mathbf{w}]$.

Lemma 2.2. *Assume that (1.4) holds and (u_ϵ, w_ϵ) is the solution of the minimization problem (2.4) for I_ϵ . Then it satisfies the equations*

$$\left\{ \begin{array}{l} \nabla \cdot (\mathcal{C} \nabla u_\epsilon + \mathcal{R} \nabla w_\epsilon) = \nabla \cdot (\mathcal{K} \nabla w_\epsilon + \mathcal{R}^T \nabla u_\epsilon) = 0 \text{ in } \Omega_\epsilon, \\ \left(\mathcal{C} \left(\sum_{k=1}^N \mathcal{U}_i + \nabla u_\epsilon \right) + \mathcal{R} \left(\sum_{k=1}^N \mathcal{W}_i + \nabla w_\epsilon \right) \right) \cdot n = 0 \text{ on } \partial \Omega, \\ \left(\mathcal{K} \left(\sum_{k=1}^N \mathcal{W}_i + \nabla w_\epsilon \right) + \mathcal{R}^T \left(\sum_{k=1}^N \mathcal{U}_i + \nabla u_\epsilon \right) \right) \cdot n = 0 \text{ on } \partial \Omega, \\ \left(\mathcal{C} \left(\sum_{j \neq i} \mathcal{U}_i + \nabla u_\epsilon \right) + \mathcal{R} \left(\sum_{j \neq i} \mathcal{W}_i + \nabla w_\epsilon \right) \right) \cdot n = 0 \text{ on } \partial B_\epsilon(\vec{d}_i), \\ \left(\mathcal{K} \left(\sum_{j \neq i} \mathcal{W}_i + \nabla w_\epsilon \right) + \mathcal{R}^T \left(\sum_{j \neq i} \mathcal{U}_i + \nabla u_\epsilon \right) \right) \cdot n = 0 \text{ on } \partial B_\epsilon(\vec{d}_i). \end{array} \right. \quad (2.5)$$

Moreover, the solution to (2.5) is unique.

Proof. This follows a standard argument via first-order variation. Letting

$$\begin{aligned} \delta I_\epsilon[\mathbf{u}, \mathbf{w}](p, q) &= \lim_{\theta \rightarrow 0} \frac{I_\epsilon[\mathbf{u} + \theta p, \mathbf{w} + \theta q] - I_\epsilon[\mathbf{u}, \mathbf{w}]}{\theta} \\ &= - \int_{\Omega_\epsilon} \left(p \nabla \cdot (\mathcal{C} \nabla \mathbf{u} + \mathcal{R} \nabla \mathbf{w}) + q \nabla \cdot (\mathcal{K} \nabla \mathbf{w} + \mathcal{R}^T \nabla \mathbf{u}) \right) dx dy \\ &\quad + \int_{\partial \Omega} p \left(\mathcal{C} \left(\sum_{k=1}^N \mathcal{U}_i + \nabla \mathbf{u} \right) + \mathcal{R} \left(\sum_{k=1}^N \mathcal{W}_i + \nabla \mathbf{w} \right) \right) \cdot n ds \\ &\quad + \int_{\partial \Omega} q \left(\mathcal{K} \left(\sum_{k=1}^N \mathcal{W}_i + \nabla \mathbf{w} \right) + \mathcal{R}^T \left(\sum_{k=1}^N \mathcal{U}_i + \nabla \mathbf{u} \right) \right) \cdot n ds \\ &\quad - \int_{\partial B_\epsilon(\vec{d}_i)} p \left(\mathcal{C} \left(\sum_{j \neq i} \mathcal{U}_i + \nabla \mathbf{u} \right) + \mathcal{R} \left(\sum_{j \neq i} \mathcal{W}_i + \nabla \mathbf{w} \right) \right) \cdot n ds \\ &\quad - \int_{\partial B_\epsilon(\vec{d}_i)} q \left(\mathcal{K} \left(\sum_{j \neq i} \mathcal{W}_i + \nabla \mathbf{w} \right) + \mathcal{R}^T \left(\sum_{j \neq i} \mathcal{U}_i + \nabla \mathbf{u} \right) \right) \cdot n ds. \end{aligned}$$

If $\delta I_\epsilon[\mathbf{u}, \mathbf{w}](p, q) = 0$ for any $p, q \in H^1(\Omega_\epsilon)$, then the system (2.5) is satisfied. The uniqueness follows from a standard argument as in the proof of Lemma 2.1. \square

2.4. Minimization of the Energy.

Lemma 2.3. *Assume that (1.4) holds. There exist constants $C_1, C_2 > 0$ independent of ϵ such that*

$$I_\epsilon[\mathbf{u}, \mathbf{w}] \geq C_1 \left(\|\mathbf{u}\|_{H^1(\Omega_\epsilon)}^2 + \|\mathbf{w}\|_{H^1(\Omega_\epsilon)}^2 \right) - C_2 \left(\|\mathbf{u}\|_{H^1(\Omega_\epsilon)} + \|\mathbf{w}\|_{H^1(\Omega_\epsilon)} \right),$$

for all $\mathbf{u}, \mathbf{w} \in H^1(\Omega_\epsilon)$ subject to the normalization condition $\int_B \mathbf{u} dx dy = 0$ and $\int_B \mathbf{w} dx dy = 0$ for some ball $B \subset \Omega_\epsilon$. Moreover, the minimization problem (2.4) for I_ϵ admits a unique solution $(u_\epsilon, w_\epsilon) \in H^1(\Omega_\epsilon)$ satisfying

$$\|\mathbf{u}\|_{H^1(\Omega_\epsilon)}^2 + \|\mathbf{w}\|_{H^1(\Omega_\epsilon)}^2 \leq M,$$

for some constant $M > 0$ independent of ϵ .

Proof. Recall that

$$\begin{aligned} I_\epsilon[\mathbf{u}, \mathbf{w}] : &= J_\epsilon[\nabla \mathbf{u}, \nabla \mathbf{w}] + \sum_{i=1}^N \int_{\partial\Omega} \left(\mathbf{u}(C\mathcal{U}_i + R\mathcal{W}_i) + \mathbf{w}(K\mathcal{W}_i + R\mathcal{U}_i) \right) \cdot n \, ds \\ &\quad - \sum_{i=1}^N \sum_{j \neq i} \int_{\partial B_\epsilon(x_i, y_i)} \left(\mathbf{u}(C\mathcal{U}_j + R\mathcal{W}_j) + \mathbf{w}(K\mathcal{W}_j + R\mathcal{U}_j) \right) \cdot n \, ds. \end{aligned}$$

Since \mathfrak{F} is positive definite, we directly estimate

$$I_\epsilon[\mathbf{u}, \mathbf{w}] \geq C \int_{\Omega_\epsilon} \left(|\nabla \mathbf{u}|^2 + |\nabla \mathbf{w}|^2 \right) dx dy - C' \int_{\partial\Omega} \left(|\mathbf{u}| + |\mathbf{w}| \right) ds - C' \int_{\partial B_\epsilon(x_i, y_i)} \left(|\mathbf{u}| + |\mathbf{w}| \right) ds.$$

By Poincaré's inequality (see [8]), we have for $C_1 > 0$ independent of ϵ ,

$$\begin{aligned} \int_{\Omega_\epsilon} |\nabla \mathbf{u}|^2 dx dy &\geq C_1 \|\mathbf{u}\|_{H^1(\Omega_\epsilon)}^2, \\ \int_{\Omega_\epsilon} |\nabla \mathbf{w}|^2 dx dy &\geq C_1 \|\mathbf{w}\|_{H^1(\Omega_\epsilon)}^2. \end{aligned}$$

In these two estimates, the normalization condition is essential. Also, we have for $C_2 > 0$ independent of ϵ ,

$$\int_{\partial\Omega} \left(|\mathbf{u}| + |\mathbf{w}| \right) ds \leq C_2 \left(\|\mathbf{u}\|_{H^1(\Omega_\epsilon)} + \|\mathbf{w}\|_{H^1(\Omega_\epsilon)} \right), \quad (2.6)$$

$$\int_{\partial B_\epsilon(x_i, y_i)} \left(|\mathbf{u}| + |\mathbf{w}| \right) ds \leq C_2 \left(\|\mathbf{u}\|_{H^1(\Omega_\epsilon)} + \|\mathbf{w}\|_{H^1(\Omega_\epsilon)} \right). \quad (2.7)$$

Hence, the coercivity is naturally valid, i.e.

$$I_\epsilon[\mathbf{u}, \mathbf{w}] \geq C_1 \left(\|\mathbf{u}\|_{H^1(\Omega_\epsilon)}^2 + \|\mathbf{w}\|_{H^1(\Omega_\epsilon)}^2 \right) - C_2 \left(\|\mathbf{u}\|_{H^1(\Omega_\epsilon)} + \|\mathbf{w}\|_{H^1(\Omega_\epsilon)} \right).$$

Since I_ϵ is strictly convex (see [8]) and $I_\epsilon[0, 0] = 0$, the existence and uniqueness follow. \square

We have established the following result.

Theorem 2.4. *Assume that (1.4) holds. The problem (1.6) admits a unique solution*

$$\mathcal{U}_\epsilon = \sum_{i=1}^N \mathcal{U}_i + \nabla u_\epsilon, \quad \mathcal{W}_\epsilon = \sum_{i=1}^N \mathcal{W}_i + \nabla w_\epsilon,$$

where

$$\begin{aligned} \mathcal{U}_i &= \frac{b_u^i}{2\pi} \frac{1}{(x - x_i)^2 + (y - y_i)^2} \begin{pmatrix} -(y - y_i) \\ (x - x_i) \end{pmatrix}, \\ \mathcal{W}_i &= \frac{b_w^i}{2\pi} \frac{1}{(x - x_i)^2 + (y - y_i)^2} \begin{pmatrix} -(y - y_i) \\ (x - x_i) \end{pmatrix}, \end{aligned}$$

and (u_ϵ, w_ϵ) is the minimizer of

$$\begin{aligned} I_\epsilon[u_\epsilon, w_\epsilon] : &= J_\epsilon[\nabla u_\epsilon, \nabla w_\epsilon] + \sum_{i=1}^N \int_{\partial\Omega} \left(u_\epsilon(C\mathcal{U}_i + R\mathcal{W}_i) + w_\epsilon(K\mathcal{W}_i + R\mathcal{U}_i) \right) \cdot n \, ds \\ &\quad - \sum_{i=1}^N \sum_{j \neq i} \int_{\partial B_\epsilon(x_i, y_i)} \left(u_\epsilon(C\mathcal{U}_j + R\mathcal{W}_j) + w_\epsilon(K\mathcal{W}_j + R\mathcal{U}_j) \right) \cdot n \, ds. \end{aligned}$$

subject to $\int_B u_\epsilon dx dy = 0$ and $\int_B w_\epsilon dx dy = 0$ for some ball $B \subset \Omega_\epsilon$, with n the outward unit normal vector on $\partial\Omega$.

This theorems tells us the existence and uniqueness of minimizer in (1.6). The asymptotic behaviors of minimizer and energy as $\epsilon \rightarrow 0$ are left open at this stage.

3. Γ -CONVERGENCE

In this section, we use higher-order Γ -convergence to dig more information into the structure of minimizer and energy.

3.1. Weak- L^2 Compactness. Notice that for any $(\tilde{\mathfrak{U}}_\epsilon, \tilde{\mathfrak{W}}_\epsilon) \in H_0^\epsilon$, using (1.4), we have

$$\begin{aligned}
& \int_{\Omega_\epsilon} \left(C |\tilde{\mathfrak{U}}_\epsilon|^2 + K |\tilde{\mathfrak{W}}_\epsilon|^2 + 2R(\tilde{\mathfrak{U}}_\epsilon \cdot \tilde{\mathfrak{W}}_\epsilon) \right) dx dy \\
& \geq \sum_{i=1}^N \int_{B_r(\vec{d}_i) \setminus B_\epsilon(\vec{d}_i)} \left(C |\tilde{\mathfrak{U}}_\epsilon|^2 + K |\tilde{\mathfrak{W}}_\epsilon|^2 + 2R(\tilde{\mathfrak{U}}_\epsilon \cdot \tilde{\mathfrak{W}}_\epsilon) \right) dx dy \\
& \geq C_0 \sum_{i=1}^N \int_{B_r(\vec{d}_i) \setminus B_\epsilon(\vec{d}_i)} \left(|\tilde{\mathfrak{U}}_\epsilon|^2 + |\tilde{\mathfrak{W}}_\epsilon|^2 \right) dx dy = C_0 \sum_{i=1}^N \int_\epsilon^r \int_{B_\rho(\vec{d}_i)} \left(|\tilde{\mathfrak{U}}_\epsilon|^2 + |\tilde{\mathfrak{W}}_\epsilon|^2 \right) ds d\rho \\
& \geq C_0 \sum_{i=1}^N \int_\epsilon^r \frac{1}{2\pi\rho} \left(\int_{B_\rho(\vec{d}_i)} (\tilde{\mathfrak{U}}_\epsilon \cdot t + \tilde{\mathfrak{W}}_\epsilon \cdot t) ds \right)^2 d\rho = C_0 \sum_{i=1}^N \int_\epsilon^r \frac{1}{2\pi\rho} (b_u^i + b_w^i)^2 d\rho \\
& = C_0 \sum_{i=1}^N \frac{(b_u^i + b_w^i)^2}{2\pi} \ln\left(\frac{r}{\epsilon}\right).
\end{aligned}$$

Therefore, we know the energy blows up when $\epsilon \rightarrow 0$. We need a proper scaling in order to show compactness. For the minimizer $(\mathcal{U}_\epsilon, \mathcal{W}_\epsilon)$, we may directly estimate

$$\begin{aligned}
& \int_{\Omega_\epsilon} \left(C |\mathcal{U}_\epsilon|^2 + K |\mathcal{W}_\epsilon|^2 + 2R(\mathcal{U}_\epsilon \cdot \mathcal{W}_\epsilon) \right) dx dy \\
& = \sum_{i=1}^N \int_{B_r(\vec{d}_i) \setminus B_\epsilon(\vec{d}_i)} \left(C |\mathcal{U}_\epsilon|^2 + K |\mathcal{W}_\epsilon|^2 + 2R(\mathcal{U}_\epsilon \cdot \mathcal{W}_\epsilon) \right) dx dy \\
& \quad + \int_{\Omega_r} \left(C |\mathcal{U}_\epsilon|^2 + K |\mathcal{W}_\epsilon|^2 + 2R(\mathcal{U}_\epsilon \cdot \mathcal{W}_\epsilon) \right) dx dy \\
& \leq C_0 \sum_{i=1}^N \frac{C(b_u^i)^2 + K(b_w^i)^2 + 2R(b_u^i)(b_w^i)}{4\pi} \ln\left(\frac{r}{\epsilon}\right).
\end{aligned} \tag{3.1}$$

Therefore, we need to consider the scaling $\frac{1}{|\ln(\epsilon)|^{1/2}}$.

Define the functional $J_\epsilon^{(0)} : L^2(\Omega) \times L^2(\Omega) \rightarrow [0, \infty]$ by

$$J_\epsilon^{(0)}[\mathfrak{U}_\epsilon, \mathfrak{W}_\epsilon] := \begin{cases} \int_{\Omega_\epsilon} \frac{1}{2} \left(C |\mathfrak{U}_\epsilon|^2 + K |\mathfrak{W}_\epsilon|^2 + 2R(\mathfrak{U}_\epsilon \cdot \mathfrak{W}_\epsilon) \right) dx dy \\ \quad \text{if } (\mathfrak{U}_\epsilon, \mathfrak{W}_\epsilon) = \left(\frac{\tilde{\mathfrak{U}}_\epsilon}{|\ln(\epsilon)|^{1/2}}, \frac{\tilde{\mathfrak{W}}_\epsilon}{|\ln(\epsilon)|^{1/2}} \right) \text{ for some } (\tilde{\mathfrak{U}}_\epsilon, \tilde{\mathfrak{W}}_\epsilon) \in H_0^\epsilon, \\ \infty \text{ otherwise in } L^2(\Omega) \times L^2(\Omega). \end{cases}$$

Theorem 3.1. (Compactness) Assume that (1.4) holds and $(\mathfrak{U}_\epsilon, \mathfrak{W}_\epsilon) \in L^2(\Omega) \times L^2(\Omega)$ satisfy

$$\sup_{\epsilon > 0} J_\epsilon^{(0)}[\mathfrak{U}_\epsilon, \mathfrak{W}_\epsilon] \leq C_0.$$

Then there exists $v_u, v_w \in H^1(\Omega)$ such that up to the extraction of subsequence (non-relabelled),

$$(\mathbf{1}_{\Omega_\epsilon} \mathfrak{U}_\epsilon, \mathbf{1}_{\Omega_\epsilon} \mathfrak{W}_\epsilon) \rightharpoonup (\nabla v_u, \nabla v_w) \text{ in weak-} L^2 \text{ as } \epsilon \rightarrow 0.$$

Proof. We use the notation as in the definition of $J_\epsilon^{(0)}$. Using the solution of a single dislocation $(\mathcal{U}_i, \mathcal{W}_i)$ in (2.2) and (2.3), recalling the definition of H_0^ϵ , we have

$$\begin{aligned} \nabla \times \left(\tilde{\mathbf{u}}_\epsilon - \sum_{i=1}^N \mathcal{U}_i \right) &= \nabla \times \left(\tilde{\mathbf{w}}_\epsilon - \sum_{i=1}^N \mathcal{W}_i \right) = 0, \\ \int_{\partial B_\epsilon(\vec{d}_i)} \left(\tilde{\mathbf{u}}_\epsilon - \sum_{i=1}^N \mathcal{U}_i \right) \cdot t \, ds &= \int_{\partial B_\epsilon(\vec{d}_i)} \left(\tilde{\mathbf{w}}_\epsilon - \sum_{i=1}^N \mathcal{W}_i \right) \cdot t \, ds = 0. \end{aligned}$$

Therefore, using the analysis of Lemma 2.1, we obtain

$$\begin{aligned} \tilde{\mathbf{u}}_\epsilon - \sum_{i=1}^N \mathcal{U}_i &= \nabla \mathbf{u}_\epsilon, \\ \tilde{\mathbf{w}}_\epsilon - \sum_{i=1}^N \mathcal{W}_i &= \nabla \mathbf{w}_\epsilon, \end{aligned}$$

for some $\mathbf{u}_\epsilon, \mathbf{w}_\epsilon \in H^1(\Omega_\epsilon)$. Also, because of (3.1) and

$$\int_{\Omega_\epsilon} \left(C |\mathcal{U}_i|^2 + K |\mathcal{W}_i|^2 + 2R(\mathcal{U}_i \cdot \mathcal{W}_i) \right) dx dy \leq C_0 |\ln(\epsilon)|,$$

we know that

$$\int_{\Omega_\epsilon} \left(C |\nabla \mathbf{u}_\epsilon|^2 + K |\nabla \mathbf{w}_\epsilon|^2 + 2R(\nabla \mathbf{u}_\epsilon \cdot \nabla \mathbf{w}_\epsilon) \right) dx dy \leq C_0 |\ln(\epsilon)|.$$

In turn, by Poincaré's inequality, we have

$$\|\mathbf{u}_\epsilon\|_{H^1(\Omega_\epsilon)} + \|\mathbf{w}_\epsilon\|_{H^1(\Omega_\epsilon)} \leq C |\ln(\epsilon)|.$$

We can define a natural extension (see [8]) of $(\mathbf{u}_\epsilon, \mathbf{w}_\epsilon)$ from Ω_ϵ to Ω as $(\hat{\mathbf{u}}_\epsilon, \hat{\mathbf{w}}_\epsilon)$ such that

$$\|\hat{\mathbf{u}}_\epsilon\|_{H^1(\Omega)} + \|\hat{\mathbf{w}}_\epsilon\|_{H^1(\Omega)} \leq C |\ln(\epsilon)|.$$

It is easy to see that up to extracting a subsequence,

$$\left(\frac{\hat{\mathbf{u}}_\epsilon}{|\ln(\epsilon)|^{1/2}}, \frac{\hat{\mathbf{w}}_\epsilon}{|\ln(\epsilon)|^{1/2}} \right) \rightharpoonup (v_u, v_w),$$

in weak- $H^1(\Omega)$ for some $(v_u, v_w) \in H^1(\Omega)$. On the other hand, note that $\mathcal{U}_i, \mathcal{W}_i \notin L^2(\Omega)$, but $\mathcal{U}_i, \mathcal{W}_i \in L^p(\Omega)$ for any $1 \leq p < 2$, and also

$$\int_{\Omega_\epsilon} \left(|\mathcal{U}_i|^2 + |\mathcal{W}_i|^2 \right) dx dy \leq C |\ln(\epsilon)|.$$

Hence, we know that up to extracting a subsequence

$$\left(\frac{\mathbf{1}_{\Omega_\epsilon} \mathcal{U}_i}{|\ln(\epsilon)|^{1/2}}, \frac{\mathbf{1}_{\Omega_\epsilon} \mathcal{W}_i}{|\ln(\epsilon)|^{1/2}} \right) \rightharpoonup (U^i, W^i),$$

in weak- $L^2(\Omega)$, for some $U^i, W^i \in L^2(\Omega)$. Taking $\phi, \psi \in C_0^\infty(\Omega)$, we have

$$\int_{\Omega_\epsilon} \frac{\mathcal{U}_i \phi + \mathcal{W}_i \psi}{|\ln(\epsilon)|^{1/2}} dx dy \leq \frac{\|\mathcal{U}_i\|_{L^1} \|\phi\|_{L^\infty} + \|\mathcal{W}_i\|_{L^1} \|\psi\|_{L^\infty}}{|\ln(\epsilon)|^{1/2}} \leq \frac{C}{|\ln(\epsilon)|^{1/2}} \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0.$$

Therefore, we must have $U^i = W^i = 0$, i.e.

$$\left(\frac{\mathbf{1}_{\Omega_\epsilon} \mathcal{U}_i}{|\ln(\epsilon)|^{1/2}}, \frac{\mathbf{1}_{\Omega_\epsilon} \mathcal{W}_i}{|\ln(\epsilon)|^{1/2}} \right) \rightharpoonup (0, 0).$$

Thus define

$$\begin{aligned}\hat{\mathfrak{U}}_\epsilon &:= \sum_{i=1}^N \mathcal{U}_i + \nabla \hat{\mathfrak{u}}_\epsilon, \\ \hat{\mathfrak{W}}_\epsilon &:= \sum_{i=1}^N \mathcal{W}_i + \nabla \hat{\mathfrak{w}}_\epsilon.\end{aligned}$$

such that $\hat{\mathfrak{U}}_\epsilon = \tilde{\mathfrak{U}}_\epsilon$ and $\hat{\mathfrak{W}}_\epsilon = \tilde{\mathfrak{W}}_\epsilon$ in Ω_ϵ . In summary, we have shown that

$$\left(\frac{\mathbf{1}_{\Omega_\epsilon} \tilde{\mathfrak{U}}_\epsilon}{|\ln(\epsilon)|^{1/2}}, \frac{\mathbf{1}_{\Omega_\epsilon} \tilde{\mathfrak{W}}_\epsilon}{|\ln(\epsilon)|^{1/2}} \right) = \left(\frac{\mathbf{1}_{\Omega_\epsilon} \hat{\mathfrak{U}}_\epsilon}{|\ln(\epsilon)|^{1/2}}, \frac{\mathbf{1}_{\Omega_\epsilon} \hat{\mathfrak{W}}_\epsilon}{|\ln(\epsilon)|^{1/2}} \right) \rightharpoonup \left(\frac{\mathbf{1}_{\Omega_\epsilon} \nabla \hat{\mathfrak{u}}_\epsilon}{|\ln(\epsilon)|^{1/2}}, \frac{\mathbf{1}_{\Omega_\epsilon} \nabla \hat{\mathfrak{w}}_\epsilon}{|\ln(\epsilon)|^{1/2}} \right) \rightharpoonup (\nabla v_u, \nabla v_w),$$

in weak- $L^2(\Omega)$.

□

3.2. Zeroth-Order Γ -Convergence.

Theorem 3.2. (*0th-Order Γ -Convergence*) Assume that (1.4) holds. Define the functional $J_0^{(0)} : L^2(\Omega) \times L^2(\Omega) \rightarrow [0, \infty]$ as

$$J_0^{(0)}[\mathfrak{U}, \mathfrak{W}] := \begin{cases} \int_\Omega \frac{1}{2} \left(C |\nabla v_u|^2 + K |\nabla v_w|^2 + 2R(\nabla v_u \cdot \nabla v_w) \right) + \sum_{i=1}^N \frac{C(b_u^i)^2 + K(b_w^i)^2 + 2R(b_u^i)(b_w^i)}{4\pi} & \text{if } (\mathfrak{U}, \mathfrak{W}) = (\nabla v_u, \nabla v_w) \text{ for some } v_u, v_w \in H^1(\Omega), \\ \infty & \text{otherwise in } L^2(\Omega) \times L^2(\Omega). \end{cases}$$

Then

- (1) For any sequence of pairs $(\mathfrak{U}_\epsilon, \mathfrak{W}_\epsilon) \in L^2(\Omega) \times L^2(\Omega)$ such that $(\mathfrak{U}_\epsilon, \mathfrak{W}_\epsilon) \rightharpoonup (\mathfrak{U}, \mathfrak{W})$ in weak- $L^2(\Omega)$, we have $\liminf_{\epsilon \rightarrow 0} J_\epsilon^{(0)}[\mathfrak{U}_\epsilon, \mathfrak{W}_\epsilon] \geq J_0^{(0)}[\nabla v_u, \nabla v_w]$.
- (2) There exists a sequence of pairs $(\mathfrak{U}_\epsilon, \mathfrak{W}_\epsilon) \in L^2(\Omega) \times L^2(\Omega)$ such that $(\mathfrak{U}_\epsilon, \mathfrak{W}_\epsilon) \rightharpoonup (\mathfrak{U}, \mathfrak{W})$ in weak- $L^2(\Omega)$, we have $\limsup_{\epsilon \rightarrow 0} J_\epsilon^{(0)}[\mathfrak{U}_\epsilon, \mathfrak{W}_\epsilon] \leq J_0^{(0)}[\nabla v_u, \nabla v_w]$,

which means

$$J_\epsilon^{(0)}[\mathfrak{U}_\epsilon, \mathfrak{W}_\epsilon] \rightarrow J_0^{(0)}[\mathfrak{U}, \mathfrak{W}],$$

in the sense of Γ -convergence in weak- $L^2(\Omega)$

Proof. We divide the proof into two steps:

Step 1: \liminf .

Assume that $(\tilde{\mathfrak{U}}_\epsilon, \tilde{\mathfrak{W}}_\epsilon) \in H_0^\epsilon$, $\left(\frac{\tilde{\mathfrak{U}}_\epsilon}{|\ln(\epsilon)|^{1/2}}, \frac{\tilde{\mathfrak{W}}_\epsilon}{|\ln(\epsilon)|^{1/2}} \right) \rightharpoonup (\mathfrak{U}, \mathfrak{W})$ and $J_0^{(0)}[\nabla v_u, \nabla v_w]$ is finite. Then due to weak convergence in L^2 and quadratic \mathfrak{F} , we know $J_\epsilon^{(0)}[\tilde{\mathfrak{U}}_\epsilon, \tilde{\mathfrak{W}}_\epsilon] \leq C_0 |\ln(\epsilon)|$. Based on compactness and Theorem 3.1, we must have

$$\left(\frac{\mathbf{1}_{\Omega_\epsilon} \tilde{\mathfrak{U}}_\epsilon}{|\ln(\epsilon)|^{1/2}}, \frac{\mathbf{1}_{\Omega_\epsilon} \tilde{\mathfrak{W}}_\epsilon}{|\ln(\epsilon)|^{1/2}} \right) \rightharpoonup (\nabla v_u, \nabla v_w),$$

for some $v_u, v_w \in H^1(\Omega)$, i.e., we must have

$$(\mathfrak{U}, \mathfrak{W}) = (\nabla v_u, \nabla v_w).$$

Based on

$$\tilde{\mathfrak{U}}_\epsilon = \sum_{i=1}^N \mathcal{U}_i + \nabla \mathfrak{u}_\epsilon, \quad \tilde{\mathfrak{W}}_\epsilon = \sum_{i=1}^N \mathcal{W}_i + \nabla \mathfrak{w}_\epsilon,$$

and the fact that

$$\left(\frac{\mathcal{U}_i}{|\ln(\epsilon)|^{1/2}}, \frac{\mathcal{W}_i}{|\ln(\epsilon)|^{1/2}} \right) \rightharpoonup (0, 0) \text{ in weak-} L^2(\Omega),$$

we deduce that

$$\left(\frac{\nabla \mathbf{u}_\epsilon}{|\ln(\epsilon)|^{1/2}}, \frac{\nabla \mathbf{w}_\epsilon}{|\ln(\epsilon)|^{1/2}} \right) \rightharpoonup (\nabla v_u, \nabla v_w) \text{ in weak-} L^2(\Omega).$$

Hence, we obtain

$$\left(\frac{\mathbf{u}_\epsilon}{|\ln(\epsilon)|^{1/2}}, \frac{\mathbf{w}_\epsilon}{|\ln(\epsilon)|^{1/2}} \right) \rightharpoonup (v_u, v_w) \text{ in weak-} H^1(\Omega).$$

For $r > \epsilon$, we write

$$\begin{aligned} & \frac{1}{|\ln(\epsilon)|} \int_{\Omega_\epsilon} \frac{1}{2} \left(C |\tilde{\mathbf{u}}_\epsilon|^2 + K |\tilde{\mathbf{w}}_\epsilon|^2 + 2R(\tilde{\mathbf{u}}_\epsilon \cdot \tilde{\mathbf{w}}_\epsilon) \right) dx dy \\ &= \frac{1}{|\ln(\epsilon)|} \int_{\Omega_r} \frac{1}{2} \left(C |\tilde{\mathbf{u}}_\epsilon|^2 + K |\tilde{\mathbf{w}}_\epsilon|^2 + 2R(\tilde{\mathbf{u}}_\epsilon \cdot \tilde{\mathbf{w}}_\epsilon) \right) dx dy \\ &+ \frac{1}{|\ln(\epsilon)|} \sum_{i=1}^N \int_{B_r(\vec{d}_i) \setminus B_\epsilon(\vec{d}_i)} \frac{1}{2} \left(C |\mathcal{U}_i|^2 + K |\mathcal{W}_i|^2 + 2R(\mathcal{U}_i \cdot \mathcal{W}_i) \right) dx dy \\ &+ \frac{1}{|\ln(\epsilon)|} \sum_{i=1}^N \int_{B_r(\vec{d}_i) \setminus B_\epsilon(\vec{d}_i)} \frac{1}{2} \left(C |\nabla \mathbf{u}_\epsilon|^2 + K |\nabla \mathbf{w}_\epsilon|^2 + 2R(\nabla \mathbf{u}_\epsilon \cdot \nabla \mathbf{w}_\epsilon) \right) dx dy \\ &+ \frac{1}{|\ln(\epsilon)|} \sum_{i \neq j} \int_{B_r(\vec{d}_i) \setminus B_\epsilon(\vec{d}_i)} \frac{1}{2} \left(C(\mathcal{U}_i \cdot \mathcal{U}_j) + K(\mathcal{W}_i \cdot \mathcal{W}_j) + R(\mathcal{U}_i \cdot \mathcal{W}_j) + R(\mathcal{W}_i \cdot \mathcal{U}_j) \right) dx dy \\ &+ \frac{1}{|\ln(\epsilon)|} \sum_{i \neq j} \int_{B_r(\vec{d}_i) \setminus B_\epsilon(\vec{d}_i)} \frac{1}{2} \left(C(\nabla \mathbf{u}_\epsilon \cdot \mathcal{U}_j) + K(\nabla \mathbf{w}_\epsilon \cdot \mathcal{W}_j) + R(\nabla \mathbf{u}_\epsilon \cdot \mathbf{w}_j) + R(\nabla \mathbf{w}_\epsilon \cdot \mathbf{u}_j) \right) dx dy \\ &+ \frac{1}{|\ln(\epsilon)|} \sum_{i=1}^N \int_{B_r(\vec{d}_i) \setminus B_\epsilon(\vec{d}_i)} \frac{1}{2} \left(C(\nabla \mathbf{u}_\epsilon \cdot \mathcal{U}_i) + K(\nabla \mathbf{w}_\epsilon \cdot \mathcal{W}_i) + R(\nabla \mathbf{u}_\epsilon \cdot \mathbf{w}_i) + R(\nabla \mathbf{w}_\epsilon \cdot \mathbf{u}_i) \right) dx dy \\ &:= I + II + III + IV + V + VI. \end{aligned}$$

By weak lower semi-continuity, we always have

$$\begin{aligned} \liminf_{\epsilon \rightarrow 0} I &\geq \int_{\Omega_r} \frac{1}{2} \left(C |\nabla v_u|^2 + K |\nabla v_w|^2 + 2R(\nabla v_u \cdot \nabla v_w) \right) dx dy \\ &\rightarrow \int_{\Omega} \frac{1}{2} \left(C |\nabla v_u|^2 + K |\nabla v_w|^2 + 2R(\nabla v_u \cdot \nabla v_w) \right) dx dy, \end{aligned}$$

as $r \rightarrow 0$. On the other hand, a direct computation based on explicit formula (2.2) and (2.3) reveals

$$\lim_{\epsilon \rightarrow 0} II = \sum_{i=1}^N \frac{C(b_u^i)^2 + K(b_w^i)^2 + 2R(b_u^i)(b_w^i)}{4\pi},$$

It is easy to see $III \geq 0$, which means

$$\liminf_{\epsilon \rightarrow 0} III \geq 0.$$

Since $i \neq j$ in IV , then in the integral, at most one of \mathcal{U}_i or \mathcal{U}_j can contribute $|\ln(\epsilon)|^{1/2}$. A similar argument holds for \mathcal{W}_i and \mathcal{W}_j . Hence, we have

$$\liminf_{\epsilon \rightarrow 0} IV = 0,$$

and

$$\liminf_{\epsilon \rightarrow 0} V = 0.$$

Since

$$\begin{cases} \nabla \cdot \mathcal{U}_i &= \nabla \cdot \mathcal{W}_i = 0 \text{ in } B_r(\vec{d}_i) \setminus B_\epsilon(\vec{d}_i), \\ \mathcal{U}_i \cdot \mathbf{n} &= \mathcal{W}_i \cdot \mathbf{n} = 0 \text{ on } \partial B_\epsilon(\vec{d}_i), \end{cases}$$

we may integrate by parts to get

$$\liminf_{\epsilon \rightarrow 0} VI = 0.$$

We have shown that

$$\liminf_{\epsilon \rightarrow 0} J_\epsilon^{(0)}[\mathfrak{U}_\epsilon, \mathfrak{W}_\epsilon] \geq J_0^{(0)}[\nabla v_u, \nabla v_w].$$

Similarly, the compactness and Theorem 3.1 imply that when $J_0^{(0)}[\nabla v_u, \nabla v_w] = \infty$, we must have $J_\epsilon^{(0)}[\mathfrak{U}_\epsilon, \mathfrak{W}_\epsilon] \rightarrow \infty$.

Step 2: lim sup.

The $J_0^{(0)}[\nabla v_u, \nabla v_w] = \infty$ case is trivial, we only consider the case when $J_0^{(0)}[\nabla v_u, \nabla v_w]$ is finite. Define

$$(\tilde{\mathfrak{U}}_\epsilon, \tilde{\mathfrak{W}}_\epsilon) := \left(|\ln(\epsilon)|^{1/2} \nabla v_u + \sum_{i=1}^N \mathcal{U}_i, |\ln(\epsilon)|^{1/2} \nabla v_w + \sum_{i=1}^N \mathcal{W}_i \right).$$

We have

$$\left[\frac{\mathbf{1}_{\Omega_\epsilon} \tilde{\mathfrak{U}}_\epsilon}{|\ln(\epsilon)|^{1/2}}, \frac{\mathbf{1}_{\Omega_\epsilon} \tilde{\mathfrak{W}}_\epsilon}{|\ln(\epsilon)|^{1/2}} \right] \rightharpoonup [\nabla v_u, \nabla v_w] \text{ in weak } -L^2(\Omega),$$

and

$$\begin{aligned} & \frac{1}{|\ln(\epsilon)|} \int_{\Omega_\epsilon} \frac{1}{2} \left(C |\tilde{\mathfrak{U}}_\epsilon|^2 + K |\tilde{\mathfrak{W}}_\epsilon|^2 + 2R(\tilde{\mathfrak{U}}_\epsilon \cdot \tilde{\mathfrak{W}}_\epsilon) \right) dx dy \\ &= \int_{\Omega_\epsilon} \frac{1}{2} \left(C |\nabla v_u|^2 + K |\nabla v_w|^2 + 2R(\nabla v_u \cdot \nabla v_w) \right) dx dy \\ &+ \frac{1}{|\ln(\epsilon)|} \sum_{i=1}^N \int_{\Omega_\epsilon} \frac{1}{2} \left(C |\mathcal{U}_i|^2 + K |\mathcal{W}_i|^2 + 2R(\mathcal{U}_i \cdot \mathcal{W}_i) \right) dx dy \\ &+ \frac{1}{|\ln(\epsilon)|} \sum_{i \neq j} \int_{\Omega_\epsilon} \frac{1}{2} \left(C(\mathcal{U}_i \cdot \mathcal{U}_j) + K(\mathcal{W}_i \cdot \mathcal{W}_j) + R(\mathcal{U}_i \cdot \mathcal{W}_j) + R(\mathcal{W}_i \cdot \mathcal{U}_j) \right) dx dy \\ &+ \frac{1}{|\ln(\epsilon)|^{1/2}} \sum_{i=1}^N \int_{\Omega_\epsilon} \frac{1}{2} \left(C(\nabla v_u \cdot \mathcal{U}_i) + K(\nabla v_w \cdot \mathcal{W}_i) + R(\nabla v_u \cdot \mathcal{W}_i) + R(\nabla v_w \cdot \mathcal{U}_i) \right) dx dy \\ &:= I + II + III + IV. \end{aligned}$$

Estimating it term by term, and using the techniques similar to those in Step 1, we have

$$\begin{aligned} \limsup_{\epsilon \rightarrow 0} I &\leq \int_{\Omega} \frac{1}{2} \left(C |\nabla v_u|^2 + K |\nabla v_w|^2 + 2R(\nabla v_u \cdot \nabla v_w) \right) dx dy, \\ \limsup_{\epsilon \rightarrow 0} II &\leq \sum_{i=1}^N \frac{C(b_u^i)^2 + K(b_w^i)^2 + 2R(b_u^i)(b_w^i)}{4\pi}, \\ \lim_{\epsilon \rightarrow 0} III &= 0, \\ \lim_{\epsilon \rightarrow 0} IV &= 0, \end{aligned}$$

and conclude that

$$\limsup_{\epsilon \rightarrow 0} J_\epsilon^{(0)}[\mathfrak{U}_\epsilon, \mathfrak{W}_\epsilon] \leq J_0^{(0)}[\mathfrak{U}, \mathfrak{W}].$$

□

By Theorem 2.4 and the basis properties of Γ -convergence, we can naturally obtain an approximation of energy.

Corollary 3.3. *Assume that (1.4) holds. We have*

$$\inf_{\mathfrak{U}, \mathfrak{W}} J_0^{(0)}[\mathfrak{U}, \mathfrak{W}] = \sum_{i=1}^N \frac{C(b_u^i)^2 + K(b_w^i)^2 + 2R(b_u^i)(b_w^i)}{4\pi}.$$

Assume $(\mathcal{W}'_\epsilon, \mathcal{W}''_\epsilon)$ is the minimizer of $J_\epsilon^{(0)}$, then we have

$$J_\epsilon^{(0)}[\mathcal{W}'_\epsilon, \mathcal{W}''_\epsilon] = E_0 + o(1),$$

where the rescaled leading-order energy

$$E_0 = \sum_{i=1}^N \frac{C(b_u^i)^2 + K(b_w^i)^2 + 2R(b_u^i)(b_w^i)}{4\pi}. \quad (3.2)$$

3.3. First-Order Γ -Convergence. Since the leading order energy E_0 only concerns with magnitude of the Burger's vectors and loses information about the dislocation position, we need more detailed analysis of convergence and selection process, which leads us to considering the first-order Γ -convergence.

Now we get rid of the rescaling $\frac{1}{|\ln(\epsilon)|^{1/2}}$. Define the functional $J_\epsilon^{(1)} : L^2(\Omega) \times L^2(\Omega) \rightarrow [0, \infty]$ as

$$J_\epsilon^{(1)}[\tilde{\mathfrak{U}}_\epsilon, \tilde{\mathfrak{W}}_\epsilon] := \begin{cases} \int_{\Omega_\epsilon} \frac{1}{2} \left(C |\tilde{\mathfrak{U}}_\epsilon|^2 + K |\tilde{\mathfrak{W}}_\epsilon|^2 + 2R(\tilde{\mathfrak{U}}_\epsilon \cdot \tilde{\mathfrak{W}}_\epsilon) \right) dx dy - |\ln(\epsilon)| \inf_{\mathfrak{U}, \mathfrak{W}} J_0^{(0)}[\mathfrak{U}, \mathfrak{W}] \\ \quad \text{if } (\tilde{\mathfrak{U}}_\epsilon, \tilde{\mathfrak{W}}_\epsilon) \in H_0^\epsilon, \\ \infty \text{ otherwise in } L^2(\Omega) \times L^2(\Omega). \end{cases}$$

Theorem 3.4. (*1st-Order Γ -Convergence*) Assume that (1.4) holds. Define the functional $J_0^{(1)} : L^2(\Omega) \times L^2(\Omega) \rightarrow [0, \infty]$ as

$$J_0^{(1)}[\tilde{\mathfrak{U}}, \tilde{\mathfrak{W}}] := \begin{cases} E_{\text{self}} + E_{\text{int}} + E_{\text{elastic}} & \text{if } (\tilde{\mathfrak{U}}, \tilde{\mathfrak{W}}) = \left(\nabla v_u + \sum_{i=1}^N \mathcal{U}_i, \nabla v_w + \sum_{i=1}^N \mathcal{W}_i \right), \\ & \text{for some } (v_u, v_w) \in H^1(\Omega), \\ \infty & \text{otherwise in } L^2(\Omega) \times L^2(\Omega), \end{cases}$$

where

$$\begin{aligned} E_{\text{self}} &:= \sum_{i=1}^N \int_{\Omega \setminus B_r(\tilde{d}_i)} \frac{1}{2} \left(C |\mathcal{U}_i|^2 + K |\mathcal{W}_i|^2 + 2R(\mathcal{U}_i \cdot \mathcal{W}_i) \right) dx dy \\ &\quad + \sum_{i=1}^N \frac{(C(b_u^i)^2 + K(b_w^i)^2 + 2R(b_u^i)(b_w^i))}{4\pi} \ln(r), \\ E_{\text{int}} &:= \sum_{i=1}^{N-1} \sum_{j=i}^N \int_{\Omega} \left(C(\mathcal{U}_i \cdot \mathcal{U}_j) + K(\mathcal{W}_i \cdot \mathcal{W}_j) + R(\mathcal{U}_i \cdot \mathcal{W}_j) + R(\mathcal{W}_j \cdot \mathcal{U}_i) \right), \\ E_{\text{elastic}} &:= J[\nabla v_u, \nabla v_w] + \sum_{i=1}^N \int_{\partial\Omega} \left(v_u(C\mathcal{U}_i + R\mathcal{W}_i) + v_w(K\mathcal{W}_i + R\mathcal{U}_i) \right) \cdot nds. \end{aligned}$$

Then

- (1) For any sequence of pairs $(\tilde{\mathfrak{U}}_\epsilon, \tilde{\mathfrak{W}}_\epsilon) \in L^2(\Omega) \times L^2(\Omega)$ such that $(\tilde{\mathfrak{U}}_\epsilon, \tilde{\mathfrak{W}}_\epsilon) \rightharpoonup (\tilde{\mathfrak{U}}, \tilde{\mathfrak{W}})$ in weak- $L^2(\Omega)$, we have $\liminf_{\epsilon \rightarrow 0} J_\epsilon^{(1)}[\tilde{\mathfrak{U}}_\epsilon, \tilde{\mathfrak{W}}_\epsilon] \geq J_0^{(1)}[\nabla v_u, \nabla v_w]$.
- (2) There exists a sequence of pairs $(\tilde{\mathfrak{U}}_\epsilon, \tilde{\mathfrak{W}}_\epsilon) \in L^2(\Omega) \times L^2(\Omega)$ such that $(\tilde{\mathfrak{U}}_\epsilon, \tilde{\mathfrak{W}}_\epsilon) \rightharpoonup (\tilde{\mathfrak{U}}, \tilde{\mathfrak{W}})$ in weak- $L^2(\Omega)$, we have $\limsup_{\epsilon \rightarrow 0} J_\epsilon^{(1)}[\tilde{\mathfrak{U}}_\epsilon, \tilde{\mathfrak{W}}_\epsilon] \leq J_0^{(1)}[\nabla v_u, \nabla v_w]$,

which means

$$J_\epsilon^{(1)}[\tilde{\mathfrak{U}}_\epsilon, \tilde{\mathfrak{W}}_\epsilon] \rightarrow J_0^{(1)}[\tilde{\mathfrak{U}}, \tilde{\mathfrak{W}}],$$

in the sense of Γ -convergence in weak- $L^2(\Omega)$

Proof. We naturally have

$$J_\epsilon^{(1)}[\tilde{\mathbf{u}}_\epsilon, \tilde{\mathbf{w}}_\epsilon] = \begin{cases} \int_{\Omega_\epsilon} \frac{1}{2} \left(C |\tilde{\mathbf{u}}_\epsilon|^2 + K |\tilde{\mathbf{w}}_\epsilon|^2 + 2R(\tilde{\mathbf{u}}_\epsilon \cdot \tilde{\mathbf{w}}_\epsilon) \right) dx dy - |\ln(\epsilon)| \sum_{i=1}^N \frac{C(b_u^i)^2 + K(b_w^i)^2 + 2R(b_u^i)(b_w^i)}{4\pi} \\ \infty & \text{otherwise in } L^2(\Omega) \times L^2(\Omega). \end{cases} \quad \text{if } (\tilde{\mathbf{u}}_\epsilon, \tilde{\mathbf{w}}_\epsilon) \in H_0^\epsilon.$$

We first prove the lim inf part. Consider weakly convergent sequence

$$(\tilde{\mathbf{u}}_\epsilon, \tilde{\mathbf{w}}_\epsilon) = \mathbf{1}_{\Omega_\epsilon} \left(\nabla \mathbf{u}_\epsilon + \sum_{i=1}^N \mathcal{U}_i, \nabla \mathbf{w}_\epsilon + \sum_{i=1}^N \mathcal{W}_i \right) \rightharpoonup (\tilde{\mathbf{u}}, \tilde{\mathbf{w}}) = \left(\nabla v_u + \sum_{i=1}^N \mathcal{U}_i, \nabla v_w + \sum_{i=1}^N \mathcal{W}_i \right),$$

Direct computation using (2.2) and (2.3) yields

$$\mathbf{1}_{\Omega_\epsilon} \left(\sum_{i=1}^N \mathcal{U}_i, \sum_{i=1}^N \mathcal{W}_i \right) \rightharpoonup \left(\sum_{i=1}^N \mathcal{U}_i, \sum_{i=1}^N \mathcal{W}_i \right).$$

Naturally, we have

$$(\nabla \mathbf{u}_\epsilon, \nabla \mathbf{w}_\epsilon) \rightharpoonup (\nabla v_u, \nabla v_w).$$

Hence, weak convergence yields boundedness $\|\nabla \mathbf{u}_\epsilon\|_{L^2(\Omega)} + \|\nabla \mathbf{w}_\epsilon\|_{L^2(\Omega)} \leq C'$ for some constant C' independent of ϵ . We may decompose

$$\begin{aligned} & \int_{\Omega_\epsilon} \frac{1}{2} \left(C |\tilde{\mathbf{u}}_\epsilon|^2 + K |\tilde{\mathbf{w}}_\epsilon|^2 + 2R(\tilde{\mathbf{u}}_\epsilon \cdot \tilde{\mathbf{w}}_\epsilon) \right) dx dy - |\ln(\epsilon)| \sum_{i=1}^N \frac{C(b_u^i)^2 + K(b_w^i)^2 + 2R(b_u^i)(b_w^i)}{4\pi} \\ &= \left(\sum_{i=1}^N \int_{\Omega_\epsilon} \frac{1}{2} \left(C |\mathcal{U}_i|^2 + K |\mathcal{W}_i|^2 + 2R(\mathcal{U}_i \cdot \mathcal{W}_i) \right) dx dy - |\ln(\epsilon)| \sum_{i=1}^N \frac{C(b_u^i)^2 + K(b_w^i)^2 + 2R(b_u^i)(b_w^i)}{4\pi} \right) \\ & \quad + \sum_{i \neq j} \int_{\Omega_\epsilon} \frac{1}{2} \left(C(\mathcal{U}_i \cdot \mathcal{U}_j) + K(\mathcal{W}_i \cdot \mathcal{W}_j) + R(\mathcal{U}_i \cdot \mathcal{W}_j) + R(\mathcal{W}_i \cdot \mathcal{U}_j) \right) dx dy \\ & \quad + \int_{\Omega_\epsilon} \frac{1}{2} \left(C |\nabla \mathbf{u}_\epsilon|^2 + K |\nabla \mathbf{w}_\epsilon|^2 + 2R(\nabla \mathbf{u}_\epsilon \cdot \nabla \mathbf{w}_\epsilon) \right) dx dy \\ & \quad + \sum_{i=1}^N \int_{\Omega_\epsilon} \frac{1}{2} \left(C(\nabla \mathbf{u}_\epsilon \cdot \mathcal{U}_i) + K(\nabla \mathbf{w}_\epsilon \cdot \mathcal{W}_i) + R(\nabla \mathbf{u}_\epsilon \cdot \mathcal{W}_i) + R(\nabla \mathbf{w}_\epsilon \cdot \mathcal{U}_i) \right) dx dy \\ &:= I + II + III + IV. \end{aligned}$$

Here the argument is similar to that in the proof of 0^{th} -order Γ -convergence, so we only describe the main strategy. For I , decompose $\Omega_\epsilon = \Omega_r \cup (\Omega_\epsilon \setminus \Omega_r)$ for some $r > \epsilon$, i.e.

$$\begin{aligned} I &= \sum_{i=1}^N \int_{\Omega_r} \frac{1}{2} \left(C |\mathcal{U}_i|^2 + K |\mathcal{W}_i|^2 + 2R(\mathcal{U}_i \cdot \mathcal{W}_i) \right) dx dy + \sum_{i=1}^N \int_{\Omega_\epsilon \setminus \Omega_r} \frac{1}{2} \left(C |\mathcal{U}_i|^2 + K |\mathcal{W}_i|^2 + 2R(\mathcal{U}_i \cdot \mathcal{W}_i) \right) dx dy \\ & \quad - |\ln(\epsilon)| \sum_{i=1}^N \frac{C(b_u^i)^2 + K(b_w^i)^2 + 2R(b_u^i)(b_w^i)}{4\pi} \end{aligned}$$

Direct computation using (2.2) and (2.3) reveals that

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \left(\sum_{i=1}^N \int_{\Omega_\epsilon \setminus \Omega_r} \frac{1}{2} \left(C |\mathcal{U}_i|^2 + K |\mathcal{W}_i|^2 + 2R(\mathcal{U}_i \cdot \mathcal{W}_i) \right) dx dy \right) \\ &= |\ln(\epsilon)| \sum_{i=1}^N \frac{C(b_u^i)^2 + K(b_w^i)^2 + 2R(b_u^i)(b_w^i)}{4\pi} + \sum_{i=1}^N \frac{C(b_u^i)^2 + K(b_w^i)^2 + 2R(b_u^i)(b_w^i)}{4\pi} \ln(r). \end{aligned}$$

Hence, we know

$$\lim_{\epsilon \rightarrow 0} I = E_{\text{self}}.$$

Similarly, a direct computation using (2.2) and (2.3) shows that

$$\lim_{\epsilon \rightarrow 0} II = E_{\text{int}}.$$

Based on weak convergence $(\nabla \mathbf{u}_\epsilon, \nabla \mathbf{w}_\epsilon) \rightharpoonup (\nabla v_u, \nabla v_w)$ and weak lower semi-continuity, we know that

$$\liminf_{\epsilon \rightarrow 0} III \geq J[\nabla v_u, \nabla v_w].$$

Finally, after integrating by parts, by weak convergence and the equations (2.1) satisfied by $(\mathcal{U}_i, \mathcal{W}_i)$, we know that

$$\lim_{\epsilon \rightarrow 0} IV = \sum_{i=1}^N \int_{\partial\Omega} \left(v_u(C\mathcal{U}_i + R\mathcal{W}_i) + v_w(K\mathcal{W}_i + R\mathcal{U}_i) \right) \cdot n ds.$$

Therefore,

$$\liminf_{\epsilon \rightarrow 0} (III + IV) \geq E_{\text{elastic}}.$$

To summarize, this concludes the proof of the \liminf part.

For the \limsup part, consider the sequence

$$(\tilde{\mathcal{U}}_\epsilon, \tilde{\mathcal{W}}_\epsilon) = \mathbf{1}_{\Omega_\epsilon} \left(\nabla v_u + \sum_{i=1}^N \mathcal{U}_i, \nabla v_w + \sum_{i=1}^N \mathcal{W}_i \right),$$

and we have

$$(\tilde{\mathcal{U}}_\epsilon, \tilde{\mathcal{W}}_\epsilon) \rightharpoonup \left(\nabla v_u + \sum_{i=1}^N \mathcal{U}_i, \nabla v_w + \sum_{i=1}^N \mathcal{W}_i \right).$$

Therefore, a direct computation using explicit formula (2.2) and (2.3) justifies the result, and thus the Γ -convergence holds. \square

Similar to the analysis of Corollary 3.3, Theorem 2.4 and the basic property of Γ -convergence justify a more detailed energy approximation.

Corollary 3.5. *Assume that (1.4) holds. We have*

$$\inf_{\tilde{\mathcal{U}}, \tilde{\mathcal{W}}} J_0^{(1)}[\tilde{\mathcal{U}}, \tilde{\mathcal{W}}] = F_{\text{self}} + F_{\text{int}} + F_{\text{elastic}},$$

where

$$\begin{aligned} F_{\text{self}} &:= \sum_{i=1}^N \int_{\Omega \setminus B_r(\vec{d}_i)} \frac{1}{2} \left(C |\mathcal{U}_i|^2 + K |\mathcal{W}_i|^2 + 2R(\mathcal{U}_i \cdot \mathcal{W}_i) \right) dx dy \\ &\quad + \sum_{i=1}^N \frac{(C(b_u^i)^2 + K(b_w^i)^2 + 2R(b_u^i)(b_w^i))}{4\pi} \ln(r), \\ F_{\text{int}} &:= \sum_{i=1}^{N-1} \sum_{j=i}^N \int_{\Omega} \left(C(\mathcal{U}_i \cdot \mathcal{U}_j) + K(\mathcal{W}_i \cdot \mathcal{W}_j) + R(\mathcal{U}_i \cdot \mathcal{W}_j) + R(\mathcal{U}_j \cdot \mathcal{W}_i) \right), \\ F_{\text{elastic}} &:= J[\nabla u_0, \nabla w_0] + \sum_{i=1}^N \int_{\partial\Omega} \left(u_0(C\mathcal{U}_i + R\mathcal{W}_i) + w_0(K\mathcal{W}_i + R\mathcal{U}_i) \right) \cdot n ds, \end{aligned} \tag{3.3}$$

in which (u_0, w_0) is the minimizer of

$$I[v_u, v_w] = J[\nabla v_u, \nabla v_w] + \sum_{i=1}^N \int_{\partial\Omega} \left(v_u(C\mathcal{U}_i + R\mathcal{W}_i) + v_w(K\mathcal{W}_i + R\mathcal{U}_i) \right) \cdot n ds.$$

Assume $(\tilde{\mathcal{U}}'_\epsilon, \tilde{\mathcal{W}}'_\epsilon) \in H_0^\epsilon$ is the minimizer of $J_\epsilon^{(1)}$, then we have

$$J_\epsilon^{(1)}[\tilde{\mathcal{U}}'_\epsilon, \tilde{\mathcal{W}}'_\epsilon] = F_{\text{self}} + F_{\text{int}} + F_{\text{elastic}} + o(1).$$

Remark 3.6. *The existence and uniqueness of minimizer (u_0, w_0) can be proved using a similar argument as in Section 2.3 and 2.4.*

Remark 3.7. *We can show that F_{self} is independent of the choice of r . Assume $r' < \bar{r}$, say $r' < r$, then we have*

$$\begin{aligned}
& \sum_{i=1}^N \int_{\Omega \setminus B_{r'}(\vec{d}_i)} \frac{1}{2} \left(C |\mathcal{U}_i|^2 + K |\mathcal{W}_i|^2 + 2R(\mathcal{U}_i \cdot \mathcal{W}_i) \right) dx dy \\
& + \sum_{i=1}^N (C(b_u^i)^2 + K(b_w^i)^2 + 2R(b_u^i)(b_w^i)) \frac{1}{4\pi} \ln(r') \\
& = \sum_{i=1}^N \int_{\Omega \setminus B_r(\vec{d}_i)} \frac{1}{2} \left(C |\mathcal{U}_i|^2 + K |\mathcal{W}_i|^2 + 2R(\mathcal{U}_i \cdot \mathcal{W}_i) \right) dx dy \\
& + \sum_{i=1}^N \int_{B_r(\vec{d}_i) \setminus B_{r'}(\vec{d}_i)} \frac{1}{2} \left(C |\mathcal{U}_i|^2 + K |\mathcal{W}_i|^2 + 2R(\mathcal{U}_i \cdot \mathcal{W}_i) \right) dx dy \\
& + \sum_{i=1}^N (C(b_u^i)^2 + K(b_w^i)^2 + 2R(b_u^i)(b_w^i)) \frac{1}{4\pi} \ln(r') \\
& = \sum_{i=1}^N \int_{\Omega \setminus B_r(\vec{d}_i)} \frac{1}{2} \left(C |\mathcal{U}_i|^2 + K |\mathcal{W}_i|^2 + 2R(\mathcal{U}_i \cdot \mathcal{W}_i) \right) dx dy \\
& + \sum_{i=1}^N (C(b_u^i)^2 + K(b_w^i)^2 + 2R(b_u^i)(b_w^i)) \frac{1}{4\pi} \ln\left(\frac{r}{r'}\right) + \sum_{i=1}^N (C(b_u^i)^2 + K(b_w^i)^2 + 2R(b_u^i)(b_w^i)) \frac{1}{4\pi} \ln(r') \\
& = \sum_{i=1}^N \int_{\Omega \setminus B_r(\vec{d}_i)} \frac{1}{2} \left(C |\mathcal{U}_i|^2 + K |\mathcal{W}_i|^2 + 2R(\mathcal{U}_i \cdot \mathcal{W}_i) \right) dx dy \\
& + \sum_{i=1}^N (C(b_u^i)^2 + K(b_w^i)^2 + 2R(b_u^i)(b_w^i)) \frac{1}{4\pi} \ln(r).
\end{aligned}$$

Hence, choosing r' or r gives exactly the same F_{self} .

3.4. Minimizer and Energy Structure. Combining Corollary 3.3 and Corollary 3.5, we can describe the structure of minimizer and energy.

Theorem 3.8. *Assume that (1.4) holds. The problem (1.6) admits a unique solution*

$$\mathcal{U}_\epsilon = \sum_{i=1}^N \mathcal{U}_i + \nabla u_\epsilon, \quad \mathcal{W}_\epsilon = \sum_{i=1}^N \mathcal{W}_i + \nabla w_\epsilon,$$

where

$$\begin{aligned}
\mathcal{U}_i &= \frac{b_u^i}{2\pi} \frac{1}{(x - x_i)^2 + (y - y_i)^2} \left(-(y - y_i), (x - x_i) \right), \\
\mathcal{W}_i &= \frac{b_w^i}{2\pi} \frac{1}{(x - x_i)^2 + (y - y_i)^2} \left(-(y - y_i), (x - x_i) \right),
\end{aligned}$$

and (u_ϵ, w_ϵ) is the unique minimizer of

$$\begin{aligned}
I_\epsilon[u_\epsilon, w_\epsilon] &:= J_\epsilon[\nabla u_\epsilon, \nabla w_\epsilon] + \sum_{i=1}^N \int_{\partial\Omega} \left(u_\epsilon(C\mathcal{U}_i + R\mathcal{W}_i) + w_\epsilon(K\mathcal{W}_i + R\mathcal{U}_i) \right) \cdot n ds \\
&\quad - \sum_{i=1}^N \sum_{j \neq i} \int_{\partial B_\epsilon(x_i, y_i)} \left(u_\epsilon(C\mathcal{U}_j + R\mathcal{W}_j) + w_\epsilon(K\mathcal{W}_j + R\mathcal{U}_j) \right) \cdot n ds,
\end{aligned}$$

subject to $\int_B u_\epsilon dx dy = 0$ and $\int_B w_\epsilon dx dy = 0$ for some ball $B \subset \Omega_\epsilon$, with n the outward unit normal vector on $\partial\Omega$.

Furthermore, $(\mathcal{U}_\epsilon, \mathcal{W}_\epsilon)$ converges in weak- $L^2(\Omega)$ as $\epsilon \rightarrow 0$ to $(\mathcal{U}_0, \mathcal{W}_0)$ where

$$\mathcal{U}_0 = \sum_{i=1}^N \mathcal{U}_i + \nabla u_0, \quad \mathcal{W}_0 = \sum_{i=1}^N \mathcal{W}_i + \nabla w_0.$$

and $[u_0, w_0]$ is the unique minimizer of

$$I_0[u_0, w_0] = J[\nabla u_0, \nabla w_0] + \sum_{i=1}^N \int_{\partial\Omega} \left(u_0(C\mathcal{U}_i + R\mathcal{W}_i) + w_0(K\mathcal{W}_i + R\mathcal{U}_i) \right) \cdot n ds,$$

subject to $\int_B u_0 dx dy = 0$ and $\int_B w_0 dx dy = 0$ for some ball $B \subset \Omega_\epsilon$

Proof. The existence and uniqueness of minimizer have been shown in Theorem 2.4. Γ -convergence naturally yields that minimizer of $J_\epsilon^{(1)}$ goes to minimizer of $J_0^{(1)}$. Hence, this result is obvious. \square

Theorem 3.9. Assume that (1.4) holds. We have

$$J_\epsilon[\mathcal{U}_\epsilon, \mathcal{W}_\epsilon] = \int_{\Omega_\epsilon} \mathfrak{F}[\mathcal{U}_\epsilon, \mathcal{W}_\epsilon] dx dy = E_0 \ln \left(\frac{1}{\epsilon} \right) + F + o(1),$$

where the core energy E_0 is defined in (3.2) and the renormalized energy $F = F_{\text{self}} + F_{\text{int}} + F_{\text{elastic}}$ is defined in (3.3).

Proof. We directly compute

$$J_\epsilon[\mathcal{U}_\epsilon, \mathcal{W}_\epsilon] = |\ln(\epsilon)| J_\epsilon^{(0)} \left[\frac{\mathcal{U}_\epsilon}{|\ln(\epsilon)|^{1/2}}, \frac{\mathcal{W}_\epsilon}{|\ln(\epsilon)|^{1/2}} \right] = E_0 \ln \left(\frac{1}{\epsilon} \right) + J_\epsilon^{(1)}[\mathcal{U}_\epsilon, \mathcal{W}_\epsilon] = E_0 \ln \left(\frac{1}{\epsilon} \right) + F + o(1).$$

\square

4. APPLICATION OF RENORMALIZED ENERGY

4.1. Interaction between Dislocations. In this section, we will prove that the energy related to interaction between dislocation F_{int} obeys the inverse logarithmical law of the distance between two dislocations.

Theorem 4.1. Assume that (1.4) holds. We have

$$F_{\text{int}} = \sum_{i=1}^{N-1} \sum_{j=i}^N \frac{Cb_u^i b_u^j + Kb_w^i b_w^j + Rb_u^i b_w^j + Rb_w^i b_u^j}{2\pi} \ln \left(\frac{1}{|\vec{d}_i - \vec{d}_j|} \right) + O(1).$$

Proof. Since

$$F_{\text{int}} = \sum_{i=1}^{N-1} \sum_{j=i}^N \int_{\Omega} \left(C(\mathcal{U}_i \cdot \mathcal{U}_j) + K(\mathcal{W}_i \cdot \mathcal{W}_j) + R(\mathcal{U}_i \cdot \mathcal{W}_j) + R(\mathcal{W}_i \cdot \mathcal{U}_j) \right),$$

let $\vec{d}_i, \vec{d}_j \in \Omega$ and let γ be a segment of line that connects \vec{d}_j to $\partial\Omega$ and is parallel to $\vec{d}_i - \vec{d}_j$. We rewrite

$$\gamma = \{ \vec{d} \in \Omega : \vec{d} = \vec{d}_j + s(\vec{d}_i - \vec{d}_j) \text{ for } s \in [0, \bar{s}] \}$$

where \bar{s} depends on the distance between \vec{d}_i, \vec{d}_j and $\partial\Omega$. Let

$$\vec{m} = \left(\frac{\vec{d}_j - \vec{d}_i}{|\vec{d}_j - \vec{d}_i|} \right)^\perp$$

indicate the unit vector perpendicular to $\vec{d}_j - \vec{d}_i$. Note that although $\Omega \setminus \{\vec{d}_j\}$ is not simply connected, $\Omega \setminus \gamma$ is. Hence, due to curl-free condition, there exist U and W such that $\mathcal{U}_j = \nabla U$ and $\mathcal{W}_j = \nabla W$ in $\Omega \setminus \gamma$ such that $[U] = b_u^j$ and $[W] = b_w^j$, where $[\cdot]$ denotes the jump across γ . By the divergence theorem, we have

$$\begin{aligned} & \int_{\Omega} \left(C(\mathcal{U}_i \cdot \mathcal{U}_j) + K(\mathcal{W}_i \cdot \mathcal{W}_j) + R(\mathcal{U}_i \cdot \mathcal{W}_j) + R(\mathcal{W}_i \cdot \mathcal{U}_j) \right) \\ &= \int_{\Omega \setminus \gamma} \left(C(\mathcal{U}_i \cdot \nabla U) + K(\mathcal{W}_i \cdot \nabla W) + R(\mathcal{U}_i \cdot \nabla W) + R(\mathcal{W}_i \cdot \nabla U) \right) \\ &= \int_{\partial\Omega} \left(C\mathcal{U}_i(U \cdot n) + K\mathcal{W}_i(W \cdot n) + R\mathcal{U}_i(W \cdot n) + R\mathcal{W}_i(U \cdot n) \right) ds \\ &\quad - \int_{\gamma} \left(C\mathcal{U}_i[U] + K\mathcal{W}_i[W] + R\mathcal{U}_i[W] + R\mathcal{W}_i[U] \right) \cdot \vec{m} ds. \end{aligned}$$

The first integral is bounded since all quantities are uniformly bounded on $\partial\Omega$. For the second integral, we estimate

$$\begin{aligned} & - \int_{\gamma} \left(C\mathcal{U}_i[U] + K\mathcal{W}_i[W] + R\mathcal{U}_i[W] + R\mathcal{W}_i[U] \right) \cdot \vec{m} ds \\ &= \int_{\gamma} \left(C\mathcal{U}_i b_u^j + K\mathcal{W}_i b_w^j + R\mathcal{U}_i b_w^j + R\mathcal{W}_i b_u^j \right) \cdot \vec{m} ds. \end{aligned}$$

By explicit formula (2.2) and (2.3), we know

$$\mathcal{U}_i(\vec{d}) = -\frac{b_u^i}{2\pi} \frac{\vec{m}}{|\vec{d} - \vec{d}_i|}, \quad \mathcal{W}_i(\vec{d}) = -\frac{b_w^i}{2\pi} \frac{\vec{m}}{|\vec{d} - \vec{d}_i|}.$$

Hence, we have

$$\begin{aligned} & \int_{\gamma} \left(C\mathcal{U}_i b_u^j + K\mathcal{W}_i b_w^j + R\mathcal{U}_i b_w^j + R\mathcal{W}_i b_u^j \right) \cdot \vec{m} ds \\ &= \int_{\gamma} \frac{Cb_u^i b_u^j + Kb_w^i b_w^j + Rb_u^i b_w^j + Rb_w^i b_u^j}{2\pi} \frac{1}{|\vec{d} - \vec{d}_i|} ds \\ &= \frac{Cb_u^i b_u^j + Kb_w^i b_w^j + Rb_u^i b_w^j + Rb_w^i b_u^j}{2\pi} \int_0^{\bar{s}} \frac{1}{|\vec{d}_i - \vec{d}_j| + s} ds \\ &= \frac{Cb_u^i b_u^j + Kb_w^i b_w^j + Rb_u^i b_w^j + Rb_w^i b_u^j}{2\pi} \left(\ln \left(\frac{1}{|\vec{d}_i - \vec{d}_j|} \right) + \ln \left(|\vec{d}_i - \vec{d}_j| + \bar{s} \right) \right). \end{aligned}$$

The result follows since we always have $\bar{s} > 0$. □

4.2. Peach-Köhler Force. The Peach-Köhler Force acting on the dislocation \vec{d}_k is given by $\nabla_{\vec{d}_k} F$ (see [10]). In this section, we will show its relation with the renormalized energy. Here we first present three lemmas proved in [8].

Lemma 4.2. *Define*

$$D_k^V f(\vec{d}) = \frac{d}{d\theta} f(\vec{d}; \vec{d}_1, \vec{d}_2, \dots, \vec{d}_k + \theta \vec{V}, \dots, \vec{d}_N) \Big|_{\theta=0}.$$

Then we have

$$\begin{aligned}
D_k^V \mathcal{U}_k &= 0 \quad \text{for } k \neq i, \\
D_k^V \mathcal{W}_k &= 0 \quad \text{for } k \neq i, \\
D_k^V \mathcal{U}_k &= -D \mathcal{U}_k \cdot \vec{V} = -\nabla(\mathcal{U}_k \cdot \vec{V}), \\
D_k^V \mathcal{W}_k &= -D \mathcal{W}_k \cdot \vec{V} = -\nabla(\mathcal{W}_k \cdot \vec{V}), \\
D_k^V \mathcal{U}_0 &= \nabla U = \nabla(D_k^V u_0 - \mathcal{U}_k \cdot \vec{V}), \\
D_k^V \mathcal{W}_0 &= \nabla W = \nabla(D_k^V w_0 - \mathcal{W}_k \cdot \vec{V})
\end{aligned}$$

where D is the derivative with respect to \vec{d} .

Lemma 4.3. *We have*

$$\begin{aligned}
\left. \frac{d}{d\theta} \int_{B_\epsilon(\vec{d}_0 + \theta \vec{V})} f(\vec{d}, \theta) dx dy \right|_{\theta=0} &= \int_{B_\epsilon(\vec{d}_0)} D_\theta f(\vec{d}, 0) dx dy \\
&= \int_{B_\epsilon(\vec{d}_0)} \partial_\theta f(\vec{d}, 0) dx dy + \int_{\partial B_\epsilon(\vec{d}_0)} f(\vec{d}, 0) \vec{V} \cdot n ds, \\
\left. \frac{d}{d\theta} \int_{\partial B_\epsilon(\vec{d}_0 + \theta \vec{V})} g(\vec{d}, \theta) ds \right|_{\theta=0} &= \int_{\partial B_\epsilon(\vec{d}_0)} D_\theta g(\vec{d}, 0) ds, \\
\left. \frac{d}{d\theta} \int_{\Omega \setminus B_\epsilon(\vec{d}_0 + \theta \vec{V})} r(\vec{d}, \theta) dx dy \right|_{\theta=0} &= \int_{\Omega \setminus B_\epsilon(\vec{d}_0)} \partial_\theta r(\vec{d}, 0) dx dy - \int_{\partial B_\epsilon(\vec{d}_0)} r(\vec{d}, 0) \vec{V} \cdot n ds,
\end{aligned}$$

where $D_\theta = \partial_\theta + \vec{V} \cdot \nabla$.

Lemma 4.4. *We have*

$$D_\theta \mathcal{U}_i(\vec{d}; \vec{d}_i + \theta \vec{V}) = 0,$$

for any \vec{V} .

Now we can prove the main result.

Theorem 4.5. *Assume that (1.4) holds. The Peach-Köhler force acting at \vec{d}_k is given by*

$$\nabla_{\vec{d}_k} F = - \int_{\partial B_r(\vec{d}_k)} \left(\mathfrak{F}[\mathcal{U}_0, \mathcal{W}_0] \mathbf{1} - (C \mathcal{U}_0 \otimes \mathcal{U}_0 + K \mathcal{W}_0 \otimes \mathcal{W}_0 + R \mathcal{U}_0 \otimes \mathcal{W}_0 + R \mathcal{W}_0 \otimes \mathcal{U}_0) \right) \cdot n ds,$$

for $r < \frac{1}{2} \min_k \left(\text{dist}(\vec{d}_k, \partial \Omega) \right)$.

Proof. We decompose the renormalized energy

$$F(\vec{d}_1, \vec{d}_2, \dots, \vec{d}_N) = G(\vec{d}_1, \vec{d}_2, \dots, \vec{d}_N) + H(\vec{d}_1, \vec{d}_2, \dots, \vec{d}_N),$$

where

$$\begin{aligned}
G(\vec{d}_1, \vec{d}_2, \dots, \vec{d}_N) &:= \int_{\Omega_\epsilon} \frac{1}{2} \left(C |\mathcal{U}_0|^2 + K |\mathcal{W}_0|^2 + 2R(\mathcal{U}_0 \cdot \mathcal{W}_0) \right) dx dy, \\
H(\vec{d}_1, \vec{d}_2, \dots, \vec{d}_N) &:= \sum_{i=1}^N \sum_{m \neq i} \int_{B_\epsilon(\vec{d}_m)} \frac{1}{2} \left(C |\mathcal{U}_i|^2 + K |\mathcal{W}_i|^2 + 2R(\mathcal{U}_i \cdot \mathcal{W}_i) \right) dx dy \\
&\quad + \sum_{m=1}^N \sum_{i=1}^{N-1} \sum_{j=i+1}^N \int_{B_\epsilon(\vec{d}_m)} \left(C(\mathcal{U}_i \cdot \mathcal{U}_j) + K(\mathcal{W}_i \cdot \mathcal{W}_j) + R(\mathcal{U}_i \cdot \mathcal{W}_j) + R(\mathcal{U}_j \cdot \mathcal{W}_i) \right) dx dy \\
&\quad + \sum_{m=1}^N \int_{B_\epsilon(\vec{d}_m)} \frac{1}{2} \left(C |\nabla u_0|^2 + K |\nabla w_0|^2 + 2R(\nabla u_0 \cdot \nabla w_0) \right) dx dy \\
&\quad + \sum_{m=1}^N \sum_{i=1}^N \int_{\partial B_\epsilon(\vec{d}_m)} \left(u_0(C \mathcal{U}_i + R \mathcal{W}_i) + w_0(K \mathcal{W}_i + R \mathcal{U}_i) \right) \cdot n ds,
\end{aligned}$$

with

$$D_k^V F = D_k^V G + D_k^V H.$$

We divide the proof into several steps:

Step 1: Estimate of $D_k^V G$.

We write

$$\begin{aligned} I &:= D_k^V \left(\int_{\Omega_\epsilon} \frac{1}{2} \left(C |\mathcal{U}_0|^2 + K |\mathcal{W}_0|^2 + 2R(\mathcal{U}_0 \cdot \mathcal{W}_0) \right) dx dy \right) \\ &= \int_{\Omega_\epsilon} \left(C \mathcal{U}_0 \cdot D_k^V \mathcal{U}_0 + K \mathcal{W}_0 \cdot D_k^V \mathcal{W}_0 + R \mathcal{U}_0 \cdot D_k^V \mathcal{W}_0 + R \mathcal{W}_0 \cdot D_k^V \mathcal{U}_0 \right) dx dy \\ &\quad - \int_{\partial B_\epsilon(\vec{d}_k)} \frac{1}{2} \left(C |\mathcal{U}_0|^2 + K |\mathcal{W}_0|^2 + 2R(\mathcal{U}_0 \cdot \mathcal{W}_0) \right) \vec{V} \cdot n ds. \end{aligned}$$

Hence, by the equations (2.1), we have

$$\begin{aligned} &\int_{\Omega_\epsilon} \left(C \mathcal{U}_0 \cdot D_k^V \mathcal{U}_0 + K \mathcal{W}_0 \cdot D_k^V \mathcal{W}_0 + R \mathcal{U}_0 \cdot D_k^V \mathcal{W}_0 + R \mathcal{W}_0 \cdot D_k^V \mathcal{U}_0 \right) dx dy \\ &= \int_{\Omega_\epsilon} \left(C \mathcal{U}_0 \cdot \nabla(D_k^V u_0 - \mathcal{U}_k \cdot \vec{V}) + K \mathcal{W}_0 \cdot \nabla(D_k^V w_0 - \mathcal{W}_k \cdot \vec{V}) \right. \\ &\quad \left. + R \mathcal{U}_0 \cdot \nabla(D_k^V w_0 - \mathcal{W}_k \cdot \vec{V}) + R \mathcal{W}_0 \cdot \nabla(D_k^V u_0 - \mathcal{U}_k \cdot \vec{V}) \right) dx dy \\ &= - \sum_{j=1}^N \int_{\partial B_\epsilon(\vec{d}_j)} \left(C \mathcal{U}_0 \cdot (D_k^V u_0 - \mathcal{U}_k \cdot \vec{V}) \cdot n + K \mathcal{W}_0 \cdot (D_k^V w_0 - \mathcal{W}_k \cdot \vec{V}) \cdot n \right. \\ &\quad \left. + R \mathcal{U}_0 \cdot (D_k^V w_0 - \mathcal{W}_k \cdot \vec{V}) \cdot n + R \mathcal{W}_0 \cdot (D_k^V u_0 - \mathcal{U}_k \cdot \vec{V}) \cdot n \right) ds. \end{aligned}$$

We obtain

$$\begin{aligned} &D_k^V \left(\int_{\Omega_\epsilon} \frac{1}{2} \left(C |\mathcal{U}_0|^2 + K |\mathcal{W}_0|^2 + 2R(\mathcal{U}_0 \cdot \mathcal{W}_0) \right) dx dy \right) \\ &= - \sum_{j=1}^N \int_{\partial B_\epsilon(\vec{d}_j)} \left(C \mathcal{U}_0 \cdot (D_k^V u_0 - \mathcal{U}_k \cdot \vec{V}) \cdot n + K \mathcal{W}_0 \cdot (D_k^V w_0 - \mathcal{W}_k \cdot \vec{V}) \cdot n \right. \\ &\quad \left. + R \mathcal{U}_0 \cdot (D_k^V w_0 - \mathcal{W}_k \cdot \vec{V}) \cdot n + R \mathcal{W}_0 \cdot (D_k^V u_0 - \mathcal{U}_k \cdot \vec{V}) \cdot n \right) ds \\ &\quad - \int_{\partial B_\epsilon(\vec{d}_k)} \frac{1}{2} \left(C |\mathcal{U}_0|^2 + K |\mathcal{W}_0|^2 + 2R(\mathcal{U}_0 \cdot \mathcal{W}_0) \right) \vec{V} \cdot n ds \\ &= - \int_{\partial B_r(\vec{d}_k)} \left(\mathfrak{F}[\mathcal{U}_0, \mathcal{W}_0] \mathbf{1} - (C \mathcal{U}_0 \otimes \mathcal{U}_0 + K \mathcal{W}_0 \otimes \mathcal{W}_0 + R \mathcal{U}_0 \otimes \mathcal{W}_0 + R \mathcal{W}_0 \otimes \mathcal{U}_0) \right) \vec{V} \cdot n ds \\ &\quad - \sum_{j \neq k} \int_{\partial B_\epsilon(\vec{d}_j)} \left(C \mathcal{U}_0 \cdot (D_k^V u_0 - \mathcal{U}_k \cdot \vec{V}) \cdot n + K \mathcal{W}_0 \cdot (D_k^V w_0 - \mathcal{W}_k \cdot \vec{V}) \cdot n \right. \\ &\quad \left. + R \mathcal{U}_0 \cdot (D_k^V w_0 - \mathcal{W}_k \cdot \vec{V}) \cdot n + R \mathcal{W}_0 \cdot (D_k^V u_0 - \mathcal{U}_k \cdot \vec{V}) \cdot n \right) ds \\ &\quad - \int_{\partial B_\epsilon(\vec{d}_k)} \left(C \mathcal{U}_0 \cdot (D_k^V u_0 - \mathcal{U}_k \cdot \vec{V}) \cdot n + K \mathcal{W}_0 \cdot (D_k^V w_0 - \mathcal{W}_k \cdot \vec{V}) \cdot n \right. \\ &\quad \left. + R \mathcal{U}_0 \cdot (D_k^V w_0 - \mathcal{W}_k \cdot \vec{V}) \cdot n + R \mathcal{W}_0 \cdot (D_k^V u_0 - \mathcal{U}_k \cdot \vec{V}) \cdot n \right. \\ &\quad \left. + (C \mathcal{U}_0 \otimes \mathcal{U}_0 + K \mathcal{W}_0 \otimes \mathcal{W}_0 + R \mathcal{U}_0 \otimes \mathcal{W}_0 + R \mathcal{W}_0 \otimes \mathcal{U}_0) \vec{V} \cdot n \right) ds \\ &= I_1 + I_2 + I_3. \end{aligned}$$

In above estimates, I_1 is the desired term, so we only focus on I_2 and I_3 . We need to cancel

$$\begin{aligned} I_2 = & - \sum_{j \neq k} \int_{\partial B_\epsilon(\vec{d}_j)} \left(C \mathcal{U}_0 \cdot (D_k^V u_0 - \mathcal{U}_k \cdot \vec{V}) \cdot n + K \mathcal{W}_0 \cdot (D_k^V w_0 - \mathcal{W}_k \cdot \vec{V}) \cdot n \right. \\ & \left. + R \mathcal{U}_0 \cdot (D_k^V w_0 - \mathcal{W}_k \cdot \vec{V}) \cdot n + R \mathcal{W}_0 \cdot (D_k^V u_0 - \mathcal{U}_k \cdot \vec{V}) \cdot n \right) ds, \end{aligned}$$

and

$$\begin{aligned} I_3 = & - \int_{\partial B_\epsilon(\vec{d}_k)} \left(C \mathcal{U}_0 \cdot (D_k^V u_0 - \mathcal{U}_k \cdot \vec{V}) \cdot n + K \mathcal{W}_0 \cdot (D_k^V w_0 - \mathcal{W}_k \cdot \vec{V}) \cdot n \right. \\ & + R \mathcal{U}_0 \cdot (D_k^V w_0 - \mathcal{W}_k \cdot \vec{V}) \cdot n + R \mathcal{W}_0 \cdot (D_k^V u_0 - \mathcal{U}_k \cdot \vec{V}) \cdot n \\ & \left. + (C \mathcal{U}_0 \otimes \mathcal{U}_0 + K \mathcal{W}_0 \otimes \mathcal{W}_0 + R \mathcal{U}_0 \otimes \mathcal{W}_0 + R \mathcal{W}_0 \otimes \mathcal{U}_0) \vec{V} \cdot n \right) ds \\ = & - \int_{\partial B_\epsilon(\vec{d}_k)} \left(C \mathcal{U}_0 \cdot (D_\theta u_0 + \sum_{j \neq k} \mathcal{U}_j \cdot \vec{V}) \cdot n + K \mathcal{W}_0 \cdot (D_\theta w_0 + \sum_{j \neq k} \mathcal{W}_j \cdot \vec{V}) \cdot n \right. \\ & \left. + R \mathcal{U}_0 \cdot (D_\theta w_0 + \sum_{j \neq k} \mathcal{W}_j \cdot \vec{V}) \cdot n + R \mathcal{W}_0 \cdot (D_\theta u_0 + \sum_{j \neq k} \mathcal{U}_j \cdot \vec{V}) \cdot n \right) ds. \end{aligned}$$

Step 2: Estimate of $D_k^V H$ - First Term.

We directly write

$$\begin{aligned} II : &= D_k^V \left(\sum_{i=1}^N \sum_{m \neq i} \int_{B_\epsilon(\vec{d}_m)} \frac{1}{2} \left(C |\mathcal{U}_i|^2 + K |\mathcal{W}_i|^2 + 2R(\mathcal{U}_i \cdot \mathcal{W}_i) \right) dx dy \right) \\ &= D_k^V \left(\sum_{m \neq k} \int_{B_\epsilon(\vec{d}_k)} \frac{1}{2} \left(C |\mathcal{U}_m|^2 + K |\mathcal{W}_m|^2 + 2R(\mathcal{U}_m \cdot \mathcal{W}_m) \right) dx dy \right) \\ &\quad + D_k^V \left(\sum_{m \neq k} \sum_{i \neq m} \int_{B_\epsilon(\vec{d}_m)} \frac{1}{2} \left(C |\mathcal{U}_i|^2 + K |\mathcal{W}_i|^2 + 2R(\mathcal{U}_i \cdot \mathcal{W}_i) \right) dx dy \right) \\ &= II_1 + II_2. \end{aligned}$$

In II_1 , we know each $D_\theta \mathcal{U}_m = D_\theta \mathcal{W}_m = 0$ since $m \neq k$, then we have

$$\begin{aligned} II_1 = & \sum_{m \neq k} \int_{B_\epsilon(\vec{d}_k)} \left(C \mathcal{U}_m \cdot \nabla(\mathcal{U}_m \cdot \vec{V}) + K \mathcal{W}_m \cdot \nabla(\mathcal{W}_m \cdot \vec{V}) \right. \\ & \left. + R \mathcal{U}_m \cdot \nabla(\mathcal{W}_m \cdot \vec{V}) + R \mathcal{W}_m \cdot \nabla(\mathcal{U}_m \cdot \vec{V}) \right) dx dy \\ = & \sum_{m \neq k} \int_{\partial B_\epsilon(\vec{d}_k)} \left(C \mathcal{U}_m \cdot n(\mathcal{U}_m \cdot \vec{V}) + K \mathcal{W}_m \cdot n(\mathcal{W}_m \cdot \vec{V}) \right. \\ & \left. + R \mathcal{U}_m \cdot n(\mathcal{W}_m \cdot \vec{V}) + R \mathcal{W}_m \cdot n(\mathcal{U}_m \cdot \vec{V}) \right) ds. \end{aligned}$$

Also, since the domain and functions do not move for $i \neq k$, we have

$$\begin{aligned}
II_2 &= D_k^V \left(\sum_{m \neq k} \int_{B_\epsilon(\vec{d}_m)} \frac{1}{2} \left(C |\mathcal{U}_k|^2 + K |\mathcal{W}_k|^2 + 2R(\mathcal{U}_k \cdot \mathcal{W}_k) \right) dx dy \right) \\
&= - \sum_{m \neq k} \int_{B_\epsilon(\vec{d}_m)} \left(C \mathcal{U}_k \cdot \nabla(\mathcal{U}_k \cdot \vec{V}) + K \mathcal{W}_k \cdot \nabla(\mathcal{W}_k \cdot \vec{V}) \right. \\
&\quad \left. + R \mathcal{U}_k \cdot \nabla(\mathcal{W}_k \cdot \vec{V}) + R \mathcal{W}_k \cdot \nabla(\mathcal{U}_k \cdot \vec{V}) \right) dx dy \\
&= - \sum_{m \neq k} \int_{\partial B_\epsilon(\vec{d}_m)} \left(C \mathcal{U}_k \cdot n(\mathcal{U}_k \cdot \vec{V}) + K \mathcal{W}_k \cdot n(\mathcal{W}_k \cdot \vec{V}) \right. \\
&\quad \left. + R \mathcal{U}_k \cdot n(\mathcal{W}_k \cdot \vec{V}) + R \mathcal{W}_k \cdot n(\mathcal{U}_k \cdot \vec{V}) \right) ds.
\end{aligned}$$

Step 3: Estimate of $D_k^V H$ - Second Term.

We directly decompose

$$\begin{aligned}
III &:= D_k^V \left(\sum_{i < j} \int_{B_\epsilon(\vec{d}_k)} \left(C(\mathcal{U}_i \cdot \mathcal{U}_j) + K(\mathcal{W}_i \cdot \mathcal{W}_j) + R(\mathcal{U}_i \cdot \mathcal{W}_j) + R(\mathcal{U}_j \cdot \mathcal{W}_i) \right) dx dy \right) \\
&\quad + D_k^V \left(\sum_{m \neq k} \sum_{i < j} \int_{B_\epsilon(\vec{d}_m)} \left(C(\mathcal{U}_i \cdot \mathcal{U}_j) + K(\mathcal{W}_i \cdot \mathcal{W}_j) + R(\mathcal{U}_i \cdot \mathcal{W}_j) + R(\mathcal{U}_j \cdot \mathcal{W}_i) \right) dx dy \right) \\
&= III_1 + III_2.
\end{aligned}$$

Then we have

$$\begin{aligned}
III_1 &= \sum_{i \neq k} \sum_{j \neq i} \int_{B_\epsilon(\vec{d}_k)} \left(C \mathcal{U}_i \cdot \nabla(\mathcal{U}_j \cdot \vec{V}) + K \mathcal{W}_i \cdot \nabla(\mathcal{W}_j \cdot \vec{V}) \right. \\
&\quad \left. + R \mathcal{U}_i \cdot \nabla(\mathcal{W}_j \cdot \vec{V}) + R \mathcal{W}_i \cdot \nabla(\mathcal{U}_j \cdot \vec{V}) \right) dx dy \\
&= \sum_{i \neq k} \sum_{j \neq i} \int_{\partial B_\epsilon(\vec{d}_k)} \left(C \mathcal{U}_i \cdot n(\mathcal{U}_j \cdot \vec{V}) + K \mathcal{W}_i \cdot n(\mathcal{W}_j \cdot \vec{V}) \right. \\
&\quad \left. + R \mathcal{U}_i \cdot n(\mathcal{W}_j \cdot \vec{V}) + R \mathcal{W}_i \cdot n(\mathcal{U}_j \cdot \vec{V}) \right) ds. \\
\\
III_2 &= - \sum_{m \neq k} \sum_{i \neq k} \int_{B_\epsilon(\vec{d}_m)} \left(C \mathcal{U}_i \cdot \nabla(\mathcal{U}_k \cdot \vec{V}) + K \mathcal{W}_i \cdot \nabla(\mathcal{W}_k \cdot \vec{V}) \right. \\
&\quad \left. + R \mathcal{U}_i \cdot \nabla(\mathcal{W}_k \cdot \vec{V}) + R \mathcal{W}_i \cdot \nabla(\mathcal{U}_k \cdot \vec{V}) \right) dx dy \\
&= - \sum_{m \neq k} \sum_{i \neq k} \int_{\partial B_\epsilon(\vec{d}_m)} \left(C \mathcal{U}_i \cdot n(\mathcal{U}_k \cdot \vec{V}) + K \mathcal{W}_i \cdot n(\mathcal{W}_k \cdot \vec{V}) \right. \\
&\quad \left. + R \mathcal{U}_i \cdot n(\mathcal{W}_k \cdot \vec{V}) + R \mathcal{W}_i \cdot n(\mathcal{U}_k \cdot \vec{V}) \right) ds.
\end{aligned}$$

Step 4: Estimate of $D_k^V H$ - Third Term.

We directly decompose

$$\begin{aligned} IV &:= D_k^V \left(\int_{B_\epsilon(\vec{d}_k)} \frac{1}{2} \left(C |\nabla u_0|^2 + K |\nabla w_0|^2 + 2R(\nabla u_0 \cdot \nabla w_0) \right) dx dy \right) \\ &\quad + D_k^V \left(\sum_{m \neq k} \int_{B_\epsilon(\vec{d}_m)} \frac{1}{2} \left(C |\nabla u_0|^2 + K |\nabla w_0|^2 + 2R(\nabla u_0 \cdot \nabla w_0) \right) dx dy \right) \\ &= IV_1 + IV_2. \end{aligned}$$

By integrating by parts, we know

$$\begin{aligned} IV_1 &= \int_{\partial B_\epsilon(\vec{d}_k)} \left(C \nabla u_0 \cdot n (D_k^V u_0 + \nabla u_0 \cdot \vec{V}) + K \nabla w_0 \cdot n (D_k^V w_0 + \nabla w_0 \cdot \vec{V}) \right. \\ &\quad \left. + R \nabla u_0 \cdot n (D_k^V w_0 + \nabla w_0 \cdot \vec{V}) + R \nabla w_0 \cdot n (D_k^V u_0 + \nabla u_0 \cdot \vec{V}) \right) ds. \end{aligned}$$

Similarly, we have

$$\begin{aligned} IV_2 &= \sum_{m \neq k} \int_{\partial B_\epsilon(\vec{d}_m)} \left(C D_k^V u_0 (\nabla u_0 \cdot n) + K D_k^V w_0 (\nabla w_0 \cdot n) \right. \\ &\quad \left. + R D_k^V w_0 (\nabla u_0 \cdot n) + R D_k^V u_0 (\nabla w_0 \cdot n) \right) ds. \end{aligned}$$

Step 5: Estimate of $D_k^V H$ - Fourth Term.

We directly decompose

$$\begin{aligned} V &:= D_k^V \left(\sum_{i=1}^N \int_{\partial B_\epsilon(\vec{d}_k)} \left(u_0 (C \mathcal{U}_i + R \mathcal{W}_i) + w_0 (K \mathcal{W}_i + R \mathcal{U}_i) \right) \cdot n ds \right) \\ &\quad + D_k^V \left(\sum_{m \neq k} \sum_{i=1}^N \int_{\partial B_\epsilon(\vec{d}_m)} \left(u_0 (C \mathcal{U}_i + R \mathcal{W}_i) + w_0 (K \mathcal{W}_i + R \mathcal{U}_i) \right) \cdot n ds \right) \\ &= V_1 + V_2. \end{aligned}$$

Similarly to previous steps, we have

$$\begin{aligned} V_1 &= \sum_{i=1}^N \int_{\partial B_\epsilon(\vec{d}_k)} \left(D_\theta u_0 (C \mathcal{U}_i + R \mathcal{W}_i) + D_\theta w_0 (K \mathcal{W}_i + R \mathcal{U}_i) \right) \cdot n ds \\ &\quad + \sum_{i \neq k} \int_{\partial B_\epsilon(\vec{d}_k)} \left(\nabla u_0 (C \mathcal{U}_i + R \mathcal{W}_i) \cdot \vec{V} + \nabla w_0 (K \mathcal{W}_i + R \mathcal{U}_i) \cdot \vec{V} \right) \cdot n ds. \end{aligned}$$

Also, we have

$$\begin{aligned} V_2 &= \sum_{m \neq k} \sum_{i=1}^N \int_{\partial B_\epsilon(\vec{d}_m)} \left(D_k^V u_0 (C \mathcal{U}_i + R \mathcal{W}_i) + D_k^V w_0 (K \mathcal{W}_i + R \mathcal{U}_i) \right) \cdot n ds \\ &\quad - \sum_{m \neq k} \sum_{i=1}^N \int_{\partial B_\epsilon(\vec{d}_k)} \left(\nabla u_0 (C \mathcal{U}_k + R \mathcal{W}_k) \cdot \vec{V} + \nabla w_0 (K \mathcal{W}_k + R \mathcal{U}_k) \cdot \vec{V} \right) \cdot n ds. \end{aligned}$$

Step 6: Synthesis.

Collecting all above terms, we have

$$\begin{aligned} II_1 + III_1 + IV_1 + V_1 &= \int_{\partial B_\epsilon(\vec{d}_k)} \left(C \mathcal{U}_0 \cdot (D_\theta u_0 + \sum_{j \neq k} \mathcal{U}_j \cdot \vec{V}) \cdot n + K \mathcal{W}_0 \cdot (D_\theta w_0 + \sum_{j \neq k} \mathcal{W}_j \cdot \vec{V}) \cdot n \right. \\ &\quad \left. + R \mathcal{U}_0 \cdot (D_\theta w_0 + \sum_{j \neq k} \mathcal{W}_j \cdot \vec{V}) \cdot n + R \mathcal{W}_0 \cdot (D_\theta u_0 + \sum_{j \neq k} \mathcal{U}_j \cdot \vec{V}) \cdot n \right) \\ &= -I_3. \end{aligned}$$

and

$$\begin{aligned}
II_2 + III_2 + IV_2 + V_2 &= \sum_{j \neq k} \int_{\partial B_\epsilon(\vec{d}_j)} \left(C\mathcal{U}_0 \cdot (D_k^V u_0 - \mathcal{U}_k \cdot \vec{V}) \cdot n + K\mathcal{W}_0 \cdot (D_k^V w_0 - \mathcal{W}_k \cdot \vec{V}) \cdot n \right. \\
&\quad \left. + R\mathcal{U}_0 \cdot (D_k^V w_0 - \mathcal{W}_k \cdot \vec{V}) \cdot n + R\mathcal{W}_0 \cdot (D_k^V u_0 - \mathcal{U}_k \cdot \vec{V}) \cdot n \right) ds \\
&= -I_2.
\end{aligned}$$

Summarizing all above, we obtain

$$\begin{aligned}
I + II + III + IV + V &= I_1 \\
&= - \int_{\partial B_r(\vec{d}_k)} \left(\mathfrak{F}[\mathcal{U}_0, \mathcal{W}_0] \mathbf{1} - (C\mathcal{U}_0 \otimes \mathcal{U}_0 + K\mathcal{W}_0 \otimes \mathcal{W}_0 + R\mathcal{U}_0 \otimes \mathcal{W}_0 + R\mathcal{W}_0 \otimes \mathcal{U}_0) \right) \vec{V} \cdot n ds.
\end{aligned}$$

Then our result naturally follows. \square

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